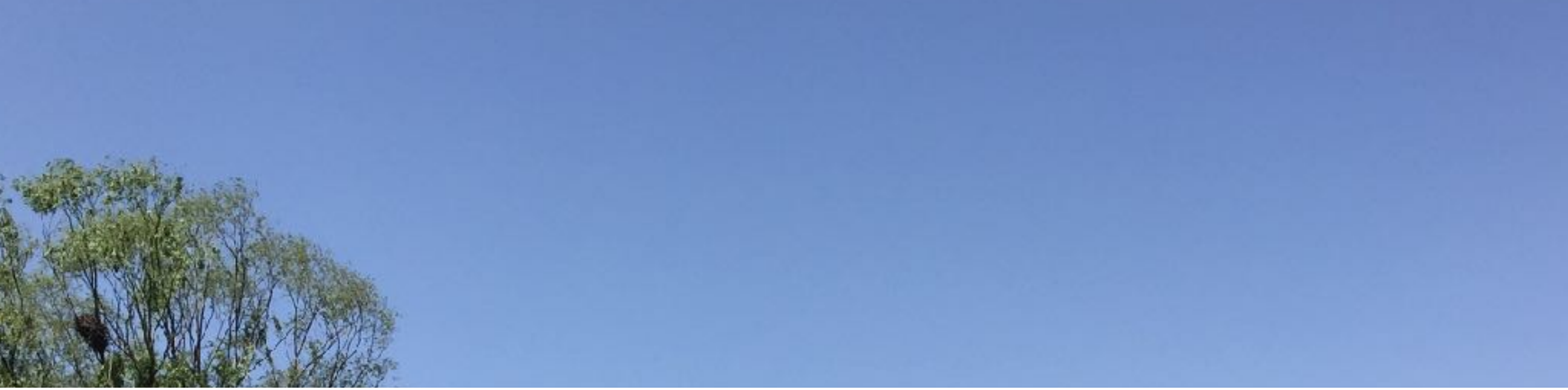

Topoi and higher-order logic

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extracts are from Maclane, Moerdijk "Sheaves in geometry and logic"

$$\mathrm{Hom}(Y \times X, Z) \to \mathrm{Hom}(Y, Z^X)$$

Definition. *In a category* $\mathbf{C}$ *with finite limits, a subobject classifier is a monic,* true: $1 \to \Omega$, *such that to every monic* $S \rightarrowtail X$ *in* $\mathbf{C}$ *there is a unique arrow* ϕ *which, with the given monic, forms a pullback square*

$$\begin{tikzcd} S \arrow[r] \arrow[d, tail] & 1 \arrow[d, tail, "\mathrm{true}"] \\ X \arrow[r, dashed, "\phi"'] & \Omega. \end{tikzcd} \tag{3}$$

Proposition 1. *A category* $\mathbf{C}$ *with finite limits and small Hom-sets has a subobject classifier if and only if there is an object* Ω *and an isomorphism*

$$\theta_X : \mathrm{Sub}_{\mathbf{C}}(X) \cong \mathrm{Hom}_{\mathbf{C}}(X, \Omega), \tag{4}$$

natural for $X \in \mathbf{C}$. *When this holds,* $\mathbf{C}$ *is well-powered.*

(i) $\mathcal{E}$ has all finite limits and colimits,
(ii) $\mathcal{E}$ has exponentials,
(iii) $\mathcal{E}$ has a subobject classifier $1 \to \Omega$.

A category $\mathcal{E}$ with these properties will be called an *elementary topos*; in brief a topos (plural: topoi). Each topos is, in particular, a cartesian closed category.

Some examples

- The category of sets
- The category of sheaves on a topological space
- The category of presheaves of sets on a small category
- The topos B of Schultz-Spivak temporal type theory

For a topos $\mathcal{E}$ we will follow much the same procedure, by regarding the objects $X, Y, \ldots$ of $\mathcal{E}$ as the "sorts" or "types" and introducing a stock of variables for each type. We thus propose to describe a "language" (called the Mitchell–Bénabou language) for $\mathcal{E}$; at the end of this section we will give a description of validity for the formulas of this language [point (4) above]. As for point (2), we will observe that the rules of inference appropriate to a general topos are precisely the standard rules for the first-order intuitionistic predicate calculus. This striking observation shows that these rules are supported by the geometrical aspects of sheaf topoi.

Finally, as in point (3), we will show that formulas $\phi(x)$ in a variable x of the Mitchell–Bénabou language can be used to specify objects of $\mathcal{E}$ by expressions of the form

$$\{x \mid \phi(x)\} \tag{1}$$

—in the fashion common in set theory. This shows how a topos behaves like a "universe of sets". By using such expressions one can, for example, mimic the usual set-theoretic constructions of the integers, rationals, reals, and complex numbers and so construct in any topos with a natural numbers object the objects of integers, rationals, reals, ...

Let us now specify the (Mitchell–Bénabou) language of a given topos $\mathcal{E}$. The types of this language are the objects of $\mathcal{E}$. We will describe the terms (expressions) of the language by recursion, beginning with the variables. For each type X there are to be variables $x, x', \dots$ of type X; each such variable has as its interpretation the identity arrow $1 \colon X \to X$. More generally, a term σ of type X will involve in its construction certain (free) variables $y, z, w, \dots$, perhaps some of them repeated. We list them in order of first occurrence, dropping any repeated variable, as y, z, w. If the respective types are Y, Z, W, then the product object $Y \times Z \times W$ in E may be called the *source* (or *domain of definition*) of the term σ, while the *interpretation* of σ is to be an arrow

$$\sigma \colon Y \times Z \times W \to X$$

Here are the inductive clauses which simultaneously define the terms of the language and their interpretation:

- Each variable x of type X is a term of type X; its interpretation is the identity $x = 1\colon X \to X$.
- Terms σ and τ of types X and Y, interpreted by $\sigma\colon U \to X$ and $\tau\colon V \to Y$, yield a term $\langle \sigma, \tau \rangle$ of type $X \times Y$; its interpretation is

$$\langle \sigma p, \tau q \rangle\colon W \to X \times Y,$$

  where the source W has evident projections $p\colon W \to U$ and $q\colon W \to V$. Here the notation $\langle \quad , \quad \rangle$ is used ambiguously, both for the new term and for the familiar map into the product $X \times Y$.
- Terms $\sigma\colon U \to X$ and $\tau\colon V \to X$ of the same type X yield a term $\sigma = \tau$ of type Ω, interpreted by

$$(\sigma = \tau)\colon W \xrightarrow{\langle \sigma p, \tau q \rangle} X \times X \xrightarrow{\quad \delta_X \quad} \Omega,$$

  where W and $\langle \sigma p, \tau q \rangle$ are as in the previous case, while δ_X is the usual characteristic map of the diagonal $\Delta\colon X \rightarrowtail X \times X$.

- An arrow $f\colon X \to Y$ of $\mathcal{E}$ and a term $\sigma\colon U \to X$ of type X together yield a term $f\circ\sigma$ of type Y, with its obvious interpretation as an actual composite

$$f\circ\sigma\colon U \xrightarrow{\sigma} X \xrightarrow{f} Y,$$

- Terms $\theta\colon V \to Y^X$ and $\sigma\colon U \to X$ of types Y^X and X yield a term $\theta(\sigma)$ of type Y interpreted by

$$\theta(\sigma)\colon W \longrightarrow Y^X \times X \xrightarrow{e} Y. \tag{2}$$

  where e is the evaluation and the map from W is $\langle \theta q, \sigma p \rangle$, much as above.
- Terms $\sigma\colon U \to X$ and $\tau\colon V \to \Omega^X$ yield a term $\sigma \in \tau$ of type Ω, interpreted as

$$\sigma \in \tau\colon W \xrightarrow{\langle \sigma p, \tau q \rangle} X \times \Omega^X \xrightarrow{e} \Omega.$$

- A variable x of type X and a term $\sigma\colon X \times U \to Z$ yield $\lambda x\sigma$, a term of type Z^X, interpreted by the transpose of σ,

$$\lambda x\sigma\colon U \to Z^X.$$

Terms $\phi, \psi, \ldots$ of type Ω will also be called *formulas* of the language. To such formulas we can apply the usual logical connectives $\wedge, \vee, \Rightarrow, \neg$, as well as the quantifiers, to get composite terms, also of type Ω. In principle, this has already been defined: the meet $\wedge\colon \Omega \times \Omega \to \Omega$ given by the internal Heyting algebra structure of Ω [see IV.6(3)] gives for terms $\phi\colon U \to \Omega$ and $\psi\colon V \to \Omega$ a new term $\wedge \circ \langle \phi, \psi \rangle\colon W \to \Omega \times \Omega \xrightarrow{\wedge} \Omega$, by the clauses above. As usual, we will denote this term more briefly as $\phi \wedge \psi$. The same procedure applies to the other propositional connectives. Thus

$$\phi \wedge \psi\colon W \xrightarrow{\langle \phi p, \psi q \rangle} \Omega \times \Omega \xrightarrow{\quad \wedge \quad} \Omega,$$

$$\phi \vee \psi\colon W \xrightarrow{\langle \phi p, \psi q \rangle} \Omega \times \Omega \xrightarrow{\quad \vee \quad} \Omega,$$

$$\phi \Rightarrow \psi\colon W \xrightarrow{\langle \phi p, \psi q \rangle} \Omega \times \Omega \xrightarrow{\quad \Rightarrow \quad} \Omega,$$

$$\neg \phi\colon W \xrightarrow{\qquad \phi \qquad} \Omega \xrightarrow{\qquad \neg \qquad} \Omega.$$

Next we interpret the quantifiers: suppose $\phi(x, y)$ is a formula containing a free variable x of type X, and others $y, \ldots$ which together give a source $X \times Y \in \mathcal{E}$ as above. Then $\phi(x, y)$ is interpreted by an arrow $X \times Y \to \Omega$ of $\mathcal{E}$. The familiar logical formalism yields a formula

$$\forall x \phi(x, y) \tag{3}$$

which no longer contains the variable x as a free variable, hence should be interpreted by an arrow $Y \to \Omega$. This can be done as follows: consider the unique map $p \colon X \to 1$, the induced map $P(p) \colon P1 \to PX$, and its internal adjoints

$$\Omega^X = PX \;\underset{\exists_p}{\overset{\forall_p}{\underset{\longrightarrow}{\overset{\longrightarrow}{\xleftarrow{P(p)}}}}}\; P1 = \Omega,$$

as in §IV.9 Theorem 2 and Proposition 4. Now the formula $\phi(x, y)$ gives a term $\lambda x \phi(x, y) \colon Y \to \Omega^X = PX$, and hence a term $\forall_p \circ \lambda x \phi(x, y) \colon Y \to \Omega$. We simply regard $\forall x \phi(x, y)$ as shorthand for $\forall_p \circ (\lambda x \phi(x, y))$. Existential formulas $\exists x \phi(x, y)$ can be treated in exactly the same way.