Spectral sequences for vertex algebras

The aim of this talk is to introduce the relevant material on spectral sequences for Ch. 15 of Frenkel & Ben-Zvi's book on vertex algebras, that is, for the quantum Drinfeld-Sokolov reduction.

The situation is that there is a bicomplex $C = C^k_0(g)$ with horizontal differential $\partial_1 = \chi$ (from the Drinfeld-Sokolov character) and vertical differential $\partial_2 = d_{sk}$ (semi-infinite cohomology of $\mathfrak{h} + (\mathfrak{h})$), which anticommute $\partial_1 \partial_2 + \partial_2 \partial_1 = 0$. That is to say,

$$C = \bigoplus_{i,j \in \mathbb{Z}} C^{i,j}, \quad \partial_1 : C^{i,j} \to C^{i+1,j}, \quad \partial_2 : C^{i,j} \to C^{i,j+1} \quad (1.1)$$

The aim is to put a vertex algebra structure on the cohomology of the total complex $(C, \partial_1 + \partial_2)$. This vertex algebra is the $W$-algebra $W_k(g)$. To compute this cohomology we use the standard spectral sequence for bicomplexes

$$H^i( H^j( C, \partial_1 ), \partial_2 ) \Rightarrow H^{i+j}( C, \partial ) \quad (1.2)$$

Note: The $\mathbb{Z}$-grading on the total complex is the obvious one

$$C^n = \bigoplus_{i+j=n} C^{i,j} \quad n = "\text{total degree}" \quad (i,j) = "\text{bidegree}"$$
In this particular case there is an isomorphism of complexes

\[(C, \partial_1) \cong (V, 0) \otimes (\tilde{C}, \tilde{\partial}_1) \tag{2.1}\]

(\text{where } V = V(a_\ast), \tilde{C} = \tilde{C}_{\mathcal{K}}(R)_{\ast} \text{ in the book}) and a filtration of \( \tilde{C} \) by subcomplexes \( \{ F_p \tilde{C} \} \), with the property that

\[\begin{align*}
0 \quad & H^\ast\left( F_p \tilde{C} / F_{p+1} \tilde{C} \right) =
\begin{cases}
C \text{ in bidegree } (0, 0) & p = 0 \\
0 & p \neq 0
\end{cases}
\end{align*}\]

\[\text{Note: } F^0 \tilde{C} \text{ is spanned by monomials of length } \leq -p \text{ applied to } \mathcal{K}.\]

We use the grading by total degree on \( \tilde{C} \), so \( H^0 (F^0 \tilde{C}) \) means elements of total deg 0 in the kernel, mod boundaries.

The next step is to deduce:

\[\begin{align*}
(2) \quad H^\ast (\tilde{C}, \tilde{\partial}_1) & \cong C \text{ using the spectral sequence associated} \\
& \text{to a filtered complex and } \mathcal{O}.
\end{align*}\]

\[\text{Note: Again, } \tilde{C}^0 \subseteq C^0 = \bigoplus_{i+j=0} C^i \tilde{j} \text{ and the generator of the} \]
\[\text{above cohomology lives in } C^0. \text{ It's important that } C^i \tilde{j} = 0 \]
\[\text{for } i < 0 \text{ and } i + j < 0, \text{ i.e. the total degree is always } \geq 0.\]
A diagram of the nonzero bidegree of $C$ is therefore

$$a_2 = 	ext{det} \quad (3.1)$$

We are told that $V \subseteq \text{Ker}(\partial_1)$ and $V \subseteq \text{C}^0$, so while elements of $V$ may have bidegree $\neq (0,0)$, they all have total degree zero.

Next, since (2.1) is an isomorphism of complexes identifying bidegrees on the left and right (this is clear from the construction),

$$H^\bullet(\text{C}, \partial_1) = V \text{ using (2) and (2.1), meaning that}$$

$$H^\bullet(\text{C}_0 \to \text{C}_1 \to \text{C}_2 \to \cdots) = V \circ 0 \circ 0 \cdots \text{ total degree zero.} \quad (3.2)$$

The spectral sequence has the left hand side as its $E_2$ page. The confusing thing is that $H^\bullet$ there refer to the (horizontal) $i$-grading only, (resp. $H_j^\bullet$), compared to the $H^\bullet$ above in (3) which is total degree.

In any case, the calculation of (3.2) shows that

$$H^i(\text{C}, \partial_1) = \bigoplus_j H^i(\cdots \to C^{i+j} \xrightarrow{\partial_1} C^{i-j} \xrightarrow{\partial_1} C^{i+j+1} \to \cdots)$$

$$= \text{Ker}(C^{i-j} \xrightarrow{\partial_1} C^{i+j-1})$$

$$= V^{c_{i-j}} \quad \text{(i.e. the bidegree} (c, -i) \text{ piece of } V \subseteq \text{C}.\)
But then clearly the induced map by $d_2$ is zero, so the $E_2$ page is identically zero, except along the diagonal $i+j=0$ where it is $V_{i,-i}$.

$$E_2$$

$$V_{i,-1} \overset{d_2}{\longrightarrow} V_{i,-2} \overset{d_2}{\longrightarrow} \cdots \overset{d_2}{\longrightarrow} V_{i,-3} \overset{d_2}{\longrightarrow} \cdots$$

Differentials on $p.2$ of a spectral sequence go across 2 and down 1, as shown, so the spectral sequence has $d_2 = 0$, i.e. it degenerates on p.2. We conclude that $\text{H}^n(C, \mathcal{A}) = 0$ unless $n = 0$, and

$$W_k(g) := \text{H}^0(C, \mathcal{A}) \cong V_{1,-1} \oplus V_{2,-2} \oplus \cdots \oplus V_{-1}.$$ (4.1)

The isomorphism $\oplus$ however is nontrivial. What the spectral sequence tells us is that the canonical filtration of $\text{H}^0(C, \mathcal{A})$ given by the images $F^p$ of the canonical maps

$$\text{H}^0(\bigoplus_{r \geq p} C^{-r}, \mathcal{A}) \overset{\text{con}}{\longrightarrow} \text{H}^0(C, \mathcal{A})$$ (4.2)

Has the property that (a) it is a decreasing filtration $F^p \succeq F^{p+1}$ and

$$F^p / F^{p+1} \cong E_{\infty}^{p,-p} \cong E_2^{p,-p} = V_{p,-p}.$$ (4.3)
Introduction to spectral sequences

Now we give a sketch of how one constructs the spectral sequences used in the above (recall: we used the SS of a filtered complex, and the SS of a bicomplex, which as it turns out is a special case of the first SS).

Our reference is:

[ thevisingtea.org / notes / Spectral Sequences.pdf ]

Let $A$ be an abelian category and $\{F^pC\}_{p\in\mathbb{Z}}$ a decreasing (i.e. $F^pC \supseteq F^{p+1}C$) chain of subcomplexes of a cochain complex $C$ in $A$. The inclusions $L_p : F^pC \rightarrow C$ induce morphisms

$$H^*(L_p) : H^*(F^pC) \rightarrow H^*(C)$$

and we define

$$E^n := H^nC, \quad F^pE^n := \text{Im}(H^n(L_p)) \subseteq E^n$$

Then clearly this gives a filtration of $E^n$ by subobjects $\{F^pE^n\}_{p\in\mathbb{Z}}$ and the question answered by the spectral sequence associated to the filtration $\{F^pC\}_{p\in\mathbb{Z}}$ is the following:

Can we “compute” the filter quotients $F^pE^n/F^{p+1}E^n$ from the filter quotients $F^pC/F^{p+1}C$?

With mild finiteness conditions on $C$, the answer is “yes”, but the nature of the “computation” is quite elaborate.
Outline

• Define $A_{r^p q^q}$, $A_{r^p q^q}$
• Define $Z_{r^p q^q}$, $B_{r^p q^q}$, $E_{r^p q^q}$
• Define $d_{r^p q^q}$
• Define $Z_{r+1} (E_{r^p q^q})$, $B_{r+1} (E_{r^p q^q})$
• Define $Z_k (E_{r^p q^q})$, $B_k (E_{r^p q^q})$ (inc $k = \infty$)
• Define $E_\infty$ and show $E_{\infty}^{p^q} \cong gr_p (E^{p^q})$
• Discuss $E_0$

• Define biregularity, first quadrant filtration
• Degeneration on page r