# Introduction to Kripke semantics

A *Kripke frame* $\mathcal{F}$ is a pair $\langle W, \leq \rangle$ where $W \neq \emptyset$ is a set of points (sometimes called 'worlds') and a partial order (reflexive, transitive, antisymmetric) $\leq \subseteq W \times W$.[1] A Kripke model is a pair $\langle \mathcal{F}, v \rangle$, where $v : W \times At \mapsto \{0, 1\}$, and $At$ is the set of atoms of the language. It is required that $v$ obeys the heredity condition:

$$\forall w, \in W, \forall p \in At, v(p, w) = 1 \text{ and } w \leq u, \text{ then } v(p, u) = 1.$$

It is also required that $v(\bot, w) = 0$, for all $w \in W$.

The assignment $v$ is extended to a valuation on the language, $\Vdash$ (read 'forces') as follows. When we need to indicate the model involved, we use a

$$\begin{array}{ccc}
w \Vdash p & \text{iff} & v(p, w) = 1 \\
w \Vdash A \wedge B & \text{iff} & w \Vdash A \text{ and } w \Vdash B \\
w \Vdash A \vee B & \text{iff} & w \Vdash A \text{ or } w \Vdash B \\
w \Vdash \neg A & \text{iff} & \forall u \geq w, u \nVdash A \\
w \Vdash A \to B & \text{iff} & \forall u \geq w, \text{ if } w \Vdash A \text{ then } w \Vdash B \\
w \nVdash \bot & &
\end{array}$$

subscript, e.g. $\Vdash_M$.

The semantics has the following Persistence Property.

**Lemma 1.** *For all sentence $A$, if $w \Vdash A$ and $w \leq u$, then $u \Vdash A$.*

*Proof.* The proof is by induction on the complexity of $A$.

The base case is handled by the heredity condition.

The cases for $\wedge$ and $\vee$ are immediate from the induction hypothesis. Suppose $w \Vdash A \wedge B$. Then $w \Vdash A$ and $w \Vdash B$. By the induction hypothesis, $u \Vdash A$ and $u \Vdash B$, whence $u \Vdash A \wedge B$. The disjunction case is similar.

[1] One could make do with a pre-order instead.

Suppose $w \Vdash A \to B$, so if $y \geq w$ and $y \Vdash A$ then $y \Vdash B$. Suppose that $u \nVdash A \to B$. Then there is a $x \geq u$ such that $x \Vdash A$ and $x \nVdash B$. Since $\leq$ is transitive, $w \leq x$. Since $x \Vdash A$, it follows that $x \Vdash B$ as well, which is a contradiction. Therefore $u \Vdash A \to B$.

The negation case is left to the reader.

□

A useful way to think about Kripke models is that the points are states of information in an investigation. The accessibility relation $\leq$ provides the different possible ways the investigation could develop. As it goes on, one builds up more and more information (true atoms). One can then assert $\neg A$ when one has excluded $A$ from all future developments.

Conjunction and disjunction are sometimes called *extensional*. Their evaluation does not change the point of evaluation. Negation and the conditional are *intensional*. Their evaluation requires evaluating their parts at other points. Any point that is unrelated to any other point acts as a classical valuation. Models with only one world reduce to classical truth tables. The classical connectives are extensional.

Kripke models are used in the study of other intensional logics, such as classical modal logic, i.e. classical logic augmented with unary operators $\Box$ and $\Diamond$ for necessity and possibility (or obligation and permission, or ...).

There is a rich collection of techniques for generating new frames and models drawn from the study of Kripke models for modal logic. One can take generated subframes, which are the frames obtained by letting $W' = \{u : w \leq u\}$, for some $w$, and restricting $\leq$ to $W'$. In Kripke semantics for intuitionistic logic, one can restrict attention to *trees* rather than arbitrary partial orders, so one can take generated subtrees. This suggests a range of options of constructing new frames and models.

An argument is *valid*, $\Gamma \models A$ in a class of models $\mathcal{C}$, iff for all models $M \in \mathcal{C}$, $\forall w \in W_M$, if $w \Vdash B$, for each $B \in \Gamma$, then $w \Vdash A$. One can show the usual relations between $\vdash$ and $\models$.

**Theorem 1** (Soundness and Completeness)**.** *$\Gamma \vdash A$ iff $\Gamma \models A$.*

The left-to-right direction is soundness. It is proved via induction on the construction of derivations. One shows the axioms are valid and the rules preserve validity.

The right-to-left direction is completeness. The proof is more involved, and we will not cover it.

With soundness in hand, we can show certain judgments underivable.

**Example 1.** *For excluded middle, take a model with* $W = \{w_1, w_2, w_3\}$ *with just* $w_1 \leq w_2$ *and* $w_1 \leq w_3$*. Take the reflexive, transitive closure of that. Set* $v(p, w_1) = 0$*,* $v(p, w_2) = 1$ *and* $v(p, w_3) = 0$*. Claim:* $w_1 \not\Vdash p \vee \neg p$*.*

**Example 2.** *For double negation elimination, take the frame from the previous example, and change the valuation so that* $v(p, w_3) = 1$ *and leave everything else the same. Claim:* $w_1 \not\Vdash \neg\neg p \to p$*.*

**Example 3.** *For the missing de Morgan law,* $(\neg(A \wedge B) \not\vdash \neg A \vee \neg B)$*, take the frame from the previous example. Set* $v(p, w_1) = v(q, w_1) = 0$*. Set* $v(p, w_2) = 1$*,* $v(q, w_2) = 0$*,* $v(p, w_3) = 0$*,* $v(q, w_3) = 1$*. Claim:* $w_1 \Vdash \neg(p \wedge q)$*, but* $w_1 \not\Vdash \neg p \vee \neg q$*.*

**Example 4.** *A few more. To show that* $\neg(p \to q) \not\models p$*, let* $W = \{w_1, w_2\}$ *and* $w_1 \leq w_2$*. Set* $v(p, w_1) = v(q, w_1) = v(q, w_2) = 0$ *and* $v(p, w_2) = 1$*. Claim:* $w_1 \Vdash \neg(p \to q)$ *but* $w_1 \not\Vdash p$*.*

**Example 5.** *To show that* $(p \wedge q) \to s \not\models (p \to s) \vee (q \to s)$*, let* $W = \{w_i : 1 \leq i \leq 5\}$ *and let* $w_1 \leq w_2 \leq w_4$ *and* $w_1 \leq w_3 \leq w_5$*. Set* $v(p, w_2) = 1$*,* $v(q, w_3) = 1$*,* $v(p, w_4) = v(p, w_5) = v(q, w_4) = v(q, w_5) = 1$*, and* $v(s, w_4) = v(s, w_5) = 1$*. Let* $v$ *make atoms 0 at any world they're not explicitly stipulated as 1. Claim:* $w_1 \Vdash (p \wedge q) \to s$ *but* $w_1 \not\Vdash (p \to s) \vee (q \to s)$*.*

**Example 6.** *To show that* $p \to q \not\models \neg p \vee q$*, take the frame from Example 1 and set* $v(p, w_2) = v(q, w_2) = 1$*, and let all atoms be 0 elsewhere. Claim:* $w_1 \models p \to q$ *but* $w_1 \not\models \neg p \vee q$*.*

Intuitionistic logic has the Disjunction Property.

**Theorem 2.** *If* $\models A \vee B$*, then* $\models A$ *or* $\models B$*.*

*Proof.* Sketch. Suppose $\models A \vee B$ but $\not\models A$ and $\not\models B$. Take a countermodel $M$ for $A$ and a countermodel $M'$ for $B$. Form a new model by taking the disjoint union of the sets of points together with a new world $w$. Let $x \leq y$ iff $x, y \in W_M$ and $x \leq_M y$, or $x, y \in W_{M'}$ and $x \leq_{M'} y$, or $x = w$. Make all atoms 0 at $w$. $w \not\Vdash A$, otherwise by the Persistence Property, $x \Vdash A$ for $x \in W_M$. Similarly, $w \not\Vdash B$. But then $w \not\Vdash A \vee B$, contradicting the assumption. □

Intuitionistic logic has the Finite Model Property: If $\not\models A$, then there is a model $M$ such that $W$ is finite and there is a $w \in W, w \not\Vdash A$. This entails that the logic is decidable.

Say that a logic is its set of theorems and that logic $L$ is a subset of another logic $L'$ just in case $L \subseteq L'$. Between intuitionistic and classical logic, there are uncountably many distinct *intermediate logics*. In terms of the proof system, one obtains an intermediate logic by adding a new axiom. In terms of Kripke semantics, one obtains an intermediate logic by restricting attention to a particular class of frames. For example, take the frames in which $\leq$ is a linear order. Call this class of frames $LC$. The axiom $(A \to B) \vee (B \to A)$ is valid in the class of all models on frames in $LC$ frames,

Consider the axiom $\neg\neg A \vee \neg A$. This is valid in all models on frames that obey the condition

$$\text{If } x \leq y \text{ and } x \leq z, \text{ then } \exists u(y \leq u \text{ and } z \leq u).$$

There is an alternative semantic clause for disjunction, due to Evert Beth. Let $\mathcal{B}$ be a subset of nodes of a tree $T$. Say that $\mathcal{B}$ is *bars* $T$ just in case every path from the root to the leaves contains some $b \in \mathcal{B}$ and that $\mathcal{B}$ *bars* a node $n \in T$ just in case $\mathcal{B}$ bars the subtree generated by $n$. Change the disjunction clause to the following

$$w \Vdash A \vee B \text{ iff } \exists \mathcal{B} \subseteq W, \mathcal{B} \text{ bars } w \text{ and } \forall u \in \mathcal{B}, b \Vdash A \text{ or } b \Vdash B.$$

Keep the definition of validity the same as before. This is the Beth semantics for intuitionistic logic. Validity in Kripke semantics and in Beth semantics both generate exactly intuitionistic logic.