Introduction to Intuitionistic Logic

August 31, 2016

We deal exclusively with propositional intuitionistic logic. The language is defined as follows.

$$\phi := p \mid \bot \mid \phi \wedge \psi \mid \phi \vee \psi \mid \phi \to \psi$$

$\neg\phi := \phi \to \bot$ and $\phi \leftrightarrow \psi := (\phi \to \psi) \wedge (\psi \to \phi)$.

A *judgment* (also called a sequent) $\Gamma \vdash \phi$, where Γ is a finite set and ϕ is a formula.[1] It is read "Γ entails ϕ" or "ϕ follows from Γ".

Some notational conventions. $\{\phi, \psi\} \vdash \theta$ is written $\phi, \psi \vdash \theta$, $\Gamma \cup \Delta$ is written Γ, Δ, $\Gamma \cup \{\phi\}$ is written Γ, ϕ, and $\emptyset \vdash \phi$ is written $\vdash \phi$.

Derivations are trees whose leaves are all axioms,

$$\Gamma, \phi \vdash \phi,$$

and which is built out of the following rules.

$$\frac{\Gamma, \phi \vdash \psi}{\Gamma \vdash \phi \to \psi} \qquad \frac{\Gamma, \phi \vdash \theta \qquad \Gamma, \psi \vdash \theta \qquad \Gamma \vdash \phi \vee \psi}{\Gamma \vdash \theta}$$

$$\frac{\Gamma \vdash \phi \qquad \Gamma \vdash \phi \to \psi}{\Gamma \vdash \psi} \qquad \frac{\Gamma \vdash \phi/\psi}{\Gamma \vdash \phi \vee \psi}$$

$$\frac{\Gamma \vdash \phi \qquad \Gamma \vdash \psi}{\Gamma \vdash \phi \wedge \psi}$$

$$\frac{\Gamma \vdash \phi \wedge \psi}{\Gamma \vdash \phi/\psi} \qquad \frac{\Gamma \vdash \bot}{\Gamma \vdash \phi}$$

[1] I think that 'judgment' is more common in the literature oriented towards type theory and computer science and 'sequent' in literature oriented towards proof theory.

A judgment $\Gamma \vdash \phi$ is derivable just in case there is a derivation whose root is $\Gamma \vdash \phi$. A provable judgment is officially $\Gamma \vdash_N \phi$, but the subscript will usually be omitted.

Why expect that the $\rightarrow$-fragment will line up with lambda calculus? This presentation of intuitionistic logic makes the correspondence relatively clear. Consider the rules for constructing derivations in the simply typed lambda calculus. They closely resemble the axiom and arrow rules for the logic. To make the connection more precise, one can annotate the logical calculus with terms that are then modified using the rules.

Natural deduction is also given in a form whose nodes are just formulas, rather than judgments. This was Gentzen's original presentation and has been investigated by Prawitz, and others. The sequent form of natural deduction brings the open premises along for each step of a derivation, whereas in Gentzen's formulation one must look at the whole tree to determine the open assumptions. The latter uses, in a sense, non-local information.

In classical logic, given $\{\rightarrow, \bot\}$, one can define the other connectives. This is not the case in intuitionistic logic. One can define negation, but not conjunction and disjunction, so we need to take the latter as primitive. The intuitionistic connectives are not truth-functional. How should we understand the connectives? A gloss known as the BHK (Brouwer-Heyting-Kolmogorov) interpretation understands them in terms of constructions, which are left imprecise.

- Atoms are primitive constructions.
- There is no construction of $\bot$.
- A construction of $\phi \wedge \psi$ consists of a construction of ϕ and a construction of ψ.
- A construction of $\phi_1 \vee \phi_2$ consists of an index $i \in \{1, 2\}$ and a construction of ϕ_i.
- A construction of $\phi \rightarrow \psi$ consists of a function that transforms constructions of ϕ into constructions of ψ.

This also gives a reason to suspect that intuitionistic logic will line up with the simply typed lambda calculus. Conditionals map items of one sort, ϕ constructions, to another sort, ψ constructions.

Intuitionistic logic agrees with classical logic on a lot, although the differences are important and give content to the claim that intuitionistic logic is constructive. In intuitionistic logic, the following are invalid.

- $\phi \vee \neg\phi$
- $\neg\neg\phi \to \phi$
- $\neg(\neg\phi \wedge \neg\psi) \to \neg\phi \vee \neg\psi$
- $((\phi \to \psi) \to \phi) \to \phi$
- $(\neg\psi \to \neg\phi) \to (\phi \to \psi)$

It is instructive to see why double negation elimination should fail. We can't *show* that it is invalid yet, but we can motivate it. Suppose that one shows that $\neg\phi$ leads to $\bot$, so one shows that $\neg\neg\phi$. This is not a construction of ϕ, but rather a construction showing that the claim that $\neg\phi$ leads to absurdity. Note, however, that negations can sometimes disappear, if one is already dealing with negated formulas, since $\neg\neg\neg\phi \leftrightarrow \neg\phi$ is derivable.

The book glosses over the distinction between derivable and admissible rules. This is an important distinction in proof theory. It is not essential for present purposes, so we won't dwell on it. A rule

$$\frac{S_1 \ldots S_k}{S}$$

is *admissible* just in case if there are derivations of $S_1, \ldots, S_k$ then there is a derivation of S. The *height of a derivation* is the number of nodes in the longest path from the root to the leaves.[2] A rule

$$\frac{S_1}{S}$$

is *height-preserving admissible* just in case if there is a derivation of S_1 with height n, then there is a derivation of S with height at most n.

The proof system, as it stands, is not that user-friendly, so we will show a few rules are admissible.

Lemma 1. *The rule*

[2]This definition needs some work, but it will get tedious to define 'path'. The meaning should be clear.

$$\frac{\Gamma \vdash \phi}{\Gamma, \Delta \vdash \phi}$$

is height-preserving admissible.

Proof. The proof is by induction on the construction of the derivation. Suppose that the only step is an axiom. Then $\Gamma, \Delta \vdash \phi$ is an axiom too, as $\phi \in \Gamma$.

The inductive steps are largely similar, so we will do one. Suppose that ϕ is $\psi \vee \theta$ and comes by $\vee$I in $n + 1$ steps. There are two cases, which are similar, depending on whether ψ or θ is the conclusion. By the induction hypothesis, $\Gamma, \Delta \vdash \psi$ is derivable in at most n steps, so $\Gamma, \Delta \vdash \psi \vee \theta$ is derivable in at most $n + 1$ steps. $\square$

Lemma 2. *The rule*

$$\frac{\Gamma \vdash \phi \qquad \Delta \vdash \psi}{\Gamma, \Delta \vdash \phi \wedge \psi}$$

is admissible, and similarly for the other multipremiss rules.

Proof. Suppose $\Gamma \vdash \phi$ and $\Delta \vdash \psi$ are derivable. By the previous lemma, $\Gamma, \Delta \vdash \phi$ and $\Gamma, \Delta \vdash \psi$ are derivable, so one can apply the conjunction introduction rule to obtain the desired conclusion. $\square$

Lemma 3. *The rule*

$$\frac{\Gamma \vdash \phi \qquad \Delta, \phi \vdash \psi}{\Gamma, \Delta \vdash \psi}$$

is admissible.

Proof. Suppose $\Gamma \vdash \phi$ and $\Delta, \phi \vdash \psi$ are derivable. Then $\Delta \vdash \phi \to \psi$ is derivable. From the previous lemma, $\Gamma, \Delta \vdash \psi$ $\square$

Lemma 4. *The rule*

$$\frac{\Gamma \vdash \phi}{\Gamma \vdash \psi \to \phi}$$

is admissible.

Proof. Suppose $\Gamma \vdash \phi$ is derivable. By lemma 1, $\Gamma, \psi \vdash \phi$ is derivable, so $\Gamma \vdash \psi \to \phi$ is derivable by $\to$I. $\square$

Define $\phi[p := \psi]$ as the formula obtained by simultaneously substituting ψ for all occurrences of p in ϕ. Define $\Gamma[p := \psi]$ as $\{\gamma[p := \psi] : \gamma \in \Gamma\}$. We further require that p not be $\perp$.[3]

Lemma 5. *The rule*

$$\frac{\Gamma \vdash \phi}{\Gamma[p := \psi] \vdash \phi[p := \psi]}$$

is admissible.

Proof. The proof is by induction on the construction of the derivation. Suppose that $\Gamma \vdash \phi$ comes by an axiom. Then $\phi \in \Gamma$, so $\phi[p := \psi] \in \Gamma[p := \psi]$. Therefore $\Gamma[p := \psi] \vdash \phi[p := \psi]$ is an axiom.

The inductive steps are all similar. We will do one, the step for disjunction introduction, so the derivation ends with the following step.

$$\frac{\Gamma \vdash \phi}{\Gamma \vdash \phi \vee \theta}$$

By the induction hypothesis, $\Gamma[p := \psi] \vdash \phi[p := \psi]$ is derivable. By $\vee$I, $\Gamma[p := \psi] \vdash \phi[p := \psi] \vee \theta[p := \psi]$ is derivable, which is the desired conclusion judgment, since substitution commutes with the logical connectives. $\square$

These lemmas make the system a bit easier to use. Right now we are interested in what judgments are derivable, so we will treat all the admissible rules above as though they were rules of the system. We could, if pressed, unpack all the derivations that use them into derivations in the system using only the basic rules. We will also follow the convention of omitting proofs of derived judgments, treating them like axioms.

Let's do some formal derivations!

$$\cfrac{\cfrac{p \vdash p}{p \vdash q \to p}}{\vdash p \to (q \to p)}$$

$$\cfrac{\cfrac{p \vdash p \qquad p \to \perp \vdash p \to \perp}{p, p \to \perp \vdash \perp}}{p \vdash (p \to \perp) \to \perp}$$

[3]What would go wrong without this condition?

$$\dfrac{\dfrac{\neg\neg\neg p \vdash \neg\neg p \to \bot \qquad p \vdash \neg\neg p}{\neg\neg\neg p, p \vdash \bot}}{\neg\neg\neg p \vdash \neg p}$$

$$\dfrac{\neg p \vee q \vdash \neg p \vee q \qquad \dfrac{\dfrac{p \vdash p \qquad \neg p \vdash \neg p}{p, \neg p \vdash \bot}}{p, \neg p \vdash p \to q} \qquad \dfrac{q \vdash q}{q \vdash p \to q}}{\neg p \vee q \vdash p \to q}$$

$$\dfrac{\dfrac{\dfrac{\neg(p \vee q) \vdash \neg(p \vee q) \qquad p \vdash p \vee q}{\neg(p \vee q), p \vdash \bot}}{\neg(p \vee q) \vdash \neg p} \qquad \dfrac{\dfrac{\neg(p \vee q) \vdash \neg(p \vee q) \qquad q \vdash p \vee q}{\neg(p \vee q), q \vdash \bot}}{\neg(p \vee q) \vdash \neg q}}{\neg(p \vee q) \vdash \neg p \wedge \neg q}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\neg(p \vee \neg p) \vdash \neg(p \vee \neg p) \qquad p \vdash p \vee \neg p}{\neg(p \vee \neg p), p \vdash \bot}}{\neg(p \vee \neg p) \vdash \neg p}}{\neg(p \vee \neg p) \vdash p \vee \neg p} \qquad \neg(p \vee \neg p) \vdash \neg(p \vee \neg p)}{\neg(p \vee \neg p) \vdash \bot}}{\vdash \neg\neg(p \vee \neg p)}$$

Intuitionistic natural deduction proofs have the feature that they can be put into a *normal form*. We won't get into the details here, but a normal proof is a proof that doesn't have detours consisting of the introduction of a connective followed immediately by its elimination. An example is given by the following fragment.

$$\dfrac{\Gamma \vdash \phi \qquad \dfrac{\Gamma, \phi \vdash \psi}{\Gamma \vdash \phi \to \psi}}{\Gamma \vdash \psi}$$

The conditional $\phi \to \psi$ is introduced then eliminated immediately. One can show that a proof with detours like that can be systematically transformed into a proof with no detours. This is the Normalizaiton Theorem, the first published proof of which is due to Prawitz. The Normalization Theorem has lots of important consequences. One consequence is that a normal proof breaks into two parts: one part whose only rules are elimination rules followed by a part whose only rules are introduction rules. Using this, one can show that formulas are not derivable. One can also show that intuitionistic logic has the *Disjunction Property*: if $\vdash \phi \vee \psi$, then either $\vdash \phi$ or $\vdash \psi$.Note that classical logic does not have this property.

The relations between intuitionistic logic and classical logic are interesting. Here's one.

Glivenko's Theorem: ϕ is a classical tautology iff $\vdash \phi$ in intuitionistic logic.

One can get a natural deduction system for classical logic by adding to the intuitionistic system either $\vdash \phi \vee \neg\phi$ as an axiom or the following rule.

$$\frac{\Gamma \vdash \neg\neg\phi}{\Gamma \vdash \phi}$$

If one is working with just the arrow fragment, one can add Peirce's rule.

$$\frac{\Gamma \vdash (\phi \rightarrow \psi) \rightarrow \phi}{\Gamma \vdash \phi}$$