## Tour of well-generated triangulated categories

The theory of well-generated triangulated categories is one of Neeman's most influential contributions, and it is a theory which continues to become of increasing relevance to other areas of mathematics as the role of "unbounded" triangulated categories widens. This talk aims to give a tour of theory, emphasising Neeman's original point of view. 10/5/17

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Shortvenion D(A), A Grothendieck abelian, is always well-generated but not always compactly generated. Brown representability still holds.

Long version ① Introduction / Def<sup>\*</sup>s ② Examples ③ Analogy to Gwthendreck ab. carts.

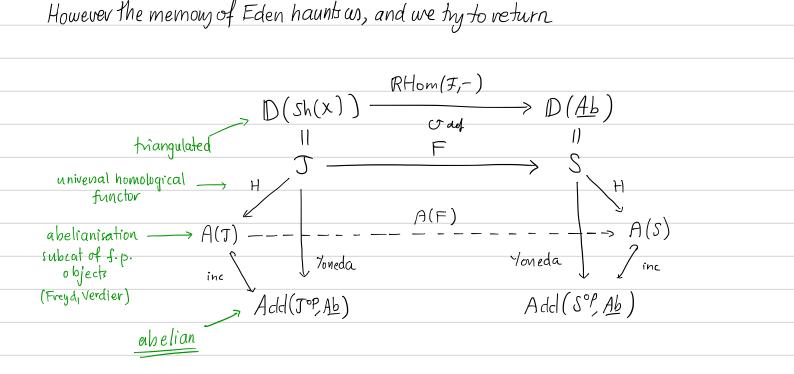
D Introduction

We begin life caring about additive function between abelian categories

 $\operatorname{Sh}(X) \xrightarrow{\operatorname{Hom}(\mathcal{F},-)} \underline{Ab}$ 

and that life is good. However at some point we are ejected from the Eden of abelian categories into the harsh world of injective resolutions and spectral sequences, ameliorated only by the language of triangulated functors between triangulated categories

 $\mathbb{D}(\operatorname{Sh}(X)) \xrightarrow{(RHom(\mathcal{F}_{r}))} \mathbb{D}(\underline{Ab}).$ 



The abelianisation hy respects all structure on J, in the sense that it sends

- · triangles to long exact sequences
- · coproducts to coproducts.

However, this A(J) is not Eden, but rather a "terrible" haunted garden:

Theorem (Neeman)  $T(-,\mathbb{Z})$  has a proper class of subobjects in A(T),  $T = \mathbb{D}(Ab)$ 

Goal Rescue the dream, by finding a good "closed subset" of A(T), 1.e.

- · a locally small abelian category CI, with
- a quotient map  $A(T) \xrightarrow{\pi} C_T$  (identifying  $C_T$  as A(T)/B, B serve and closed under pivolucts, s.t. T has a fully faithful left adjoint. Thin  $\mathbb{D}(X) \xrightarrow{c_{min}} \mathbb{D}_Z(X)$
- the composite J→ A(J)→ CJ maps nonzero objects to nonzero objects, and preserve coproclucts, and
- T→ CJ is functorial.



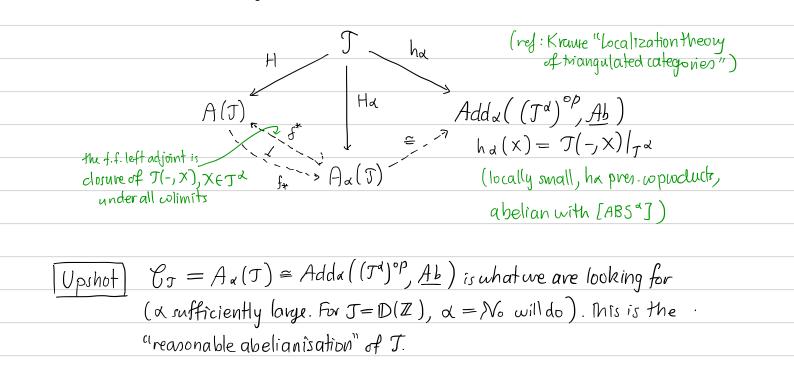
This goal is achieved by the theory of well-generated triangulated categories. Let me sketch how, before explaining definitions.

If J is well-generated then

- $T = \bigcup_{\alpha \text{ regular}} T^{\alpha}$ ,  $T^{\alpha} \alpha$ -localising (triang. subcat closed under  $\alpha \alpha$  wpwclucts) Objects of  $T^{\alpha}$  are called  $\alpha$ -compact ( $T^{N_0} = T^{c}$ ).
- An abelian category satisfies [AB5<sup>4</sup>] if it is [AB4] (wopwclucks of monos are mono) and a filtered colimits are exact.
- · For & regular there is a universal workcluct-preserving homological functor

$$H_{\alpha}: \mathcal{J} \longrightarrow \mathcal{A}_{\alpha}(\mathcal{J})$$

into an abelian category  $A_a(T)$  which is  $[AB5^{\alpha}]$  (Neeman, Appendix B).



From now on T is a thiongulated category with arbitrary copy clucts, small Hom sets, and  $\alpha$  is a regular cardinal.

$$\begin{array}{c} \underline{\operatorname{Def}}^{*} \ X \in J \text{ is } \underline{\prec} \operatorname{-\operatorname{small}} \text{ if every morphism into a coproduct} \\ & \times \longrightarrow \amalg_{i \in I} Y_{i} \\ & factor through a subcoprocluct \\ & \times \longrightarrow \amalg_{i \in T} Y_{i} \longrightarrow \amalg_{i \in I} X_{i}' \\ & \text{with } |J| < x. \text{ Let } \mathcal{T}^{(a)} = \{X \in J \mid X \text{ is } d \text{-small}\}. \\ \hline \\ \underline{\operatorname{Def}}^{A} \ A \ class \ T \subseteq J \text{ is } an \ \underline{\alpha} \text{-perfect } class \ if \ it \ contains O, \ and \\ & (i) \ Any morphism \ t \longrightarrow \amalg_{i \in I} Y_{i} \ with \ t \in T, \ Y_{i} \in J, \ |I| < x \ factors \ as \\ & \quad t \longrightarrow \coprod_{i \in I} t_{i} \ \lim_{i \in I} Y_{i} \\ & \quad for \ some \ t_{i} \in T. \\ & (ii) \ Given \ morphisms \ \{f_{i}: t_{i} \longrightarrow Y_{i}\}_{i \in I} \ with \ t \in T, \ Y_{i} \in J, \ |I| < x \ if \\ & \quad a \ composite \ of \ the \ form \\ & \quad t \longrightarrow \coprod_{i \in I} t_{i} \ \lim_{i \in I} Y_{i} \\ & \quad t \longrightarrow \coprod_{i \in I} Y_{i} \\ & \quad t \longrightarrow \coprod_{i \in I} t_{i} \ \lim_{i \in I} Y_{i} \\ & \quad t \longrightarrow \coprod_{i \in I} Y_{i} \\ & \quad t \longrightarrow \coprod_{i \in I} t_{i} \ \lim_{i \in I} Y_{i} \\ & \quad t \longrightarrow \coprod_{i \in I} t_{i} \ \lim_{i \in I} Y_{i} \\ & \quad t \longrightarrow \coprod_{i \in I} t_{i} \ \lim_{i \in I} Y_{i} \\ & \quad t \longrightarrow \coprod_{i \in I} t_{i} \ \lim_{i \in I} Y_{i} \\ & \quad t \longrightarrow \coprod_{i \in I} Y_{i} \\ & \quad t \longrightarrow \coprod_{i \in I} Y_{i} \\ & \quad t \longrightarrow \coprod_{i \in I} t_{i} \ \lim_{i \in I} Y_{i} \\ & \quad t \longrightarrow \coprod_{i \in I} t_{i} \ \lim_{i \in I} Y_{i} \\ & \quad t \longrightarrow \coprod_{i \in I} t_{i} \ \lim_{i \in I} Y_{i} \\ & \quad t \longrightarrow \coprod_{i \in I} t_{i} \ \lim_{i \in I} Y_{i} \\ & \quad t \mapsto \coprod_{i \in I} t_{i} \ \lim_{i \in I} Y_{i} \\ & \quad t \to \coprod_{i \in I} t_{i} \ \lim_{i \in I} Y_{i} \\ & \quad t \to \coprod_{i \in I} t_{i} \ \lim_{i \in I} Y_{i} \\ & \quad t \to \coprod_{i \in I} t_{i} \ \lim_{i \in I} Y_{i} \\ & \quad t \to \coprod_{i \in I} t_{i} \ \lim_{i \in I} Y_{i} \\ & \quad t \to \coprod_{i \in I} t_{i} \ \lim_{i \in I} Y_{i} \\ & \quad t \to \coprod_{i \in I} t_{i} \ \lim_{i \in I} Y_{i} \\ & \quad t \to \coprod_{i \in I} T_{i} \ \lim_{i \in I} Y_{i} \\ & \quad t \to \coprod_{i \in I} T_{i} \ \lim_{i \in I} Y_{i} \\ & \quad t \to \coprod_{i \in I} T_{i} \ H_{i} \\ & \quad t \to \coprod_{i \in I} T_{i} \ H_{i} \\ & \quad t \to \coprod_{i \in I} T_{i} \ H_{i} \\ & \quad t \to \coprod_{i \in I} T_{i} \ H_{i} \\ & \quad t \to \coprod_{i \in I} T_{i} \ H_{i} \\ & \quad t \to \coprod_{i \in I} T_{i} \ H_{i} \\ & \quad t \to \coprod_{i \in I} T_{i} \ H_{i} \\ & \quad t \to \coprod_{i \in I} T_{i} \ H_{i} \\ & \quad t \to \coprod_{i \in I} T_{i} \ H_{i} \\ & \quad t \to \coprod_{i \in I} T_{i} \ H_{i} \\ & \quad t \to \coprod_{i \in I} T_{i} \ H_{i} \\ & \quad t \to \coprod_{i \in I} T_{i} \ H_{i} \\$$

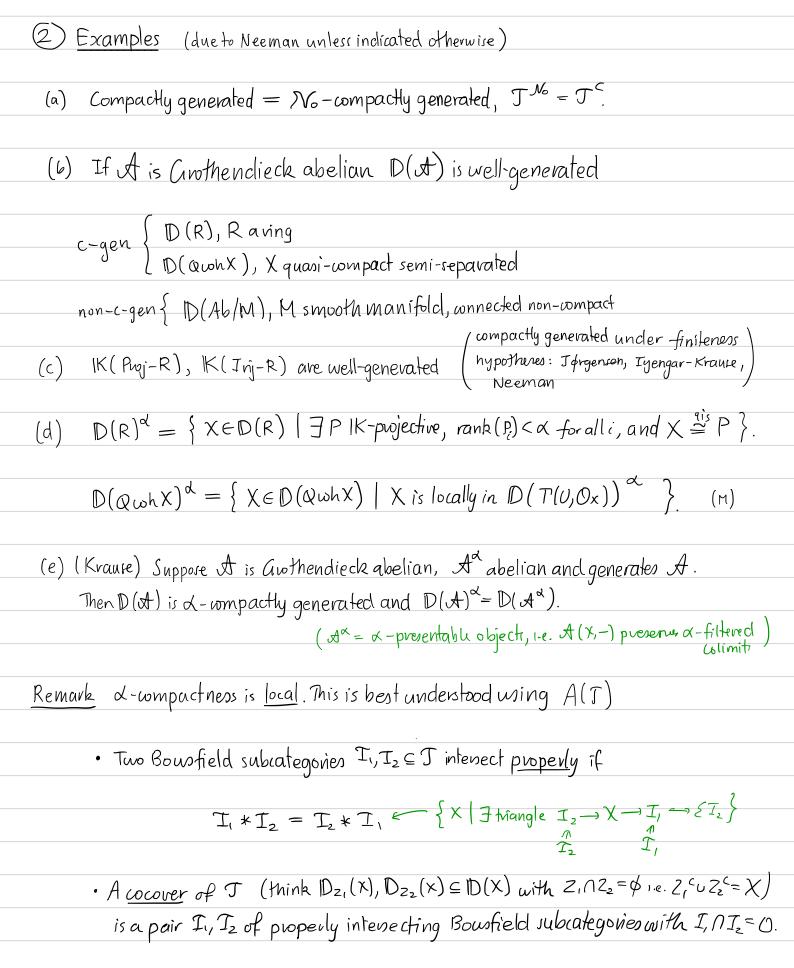


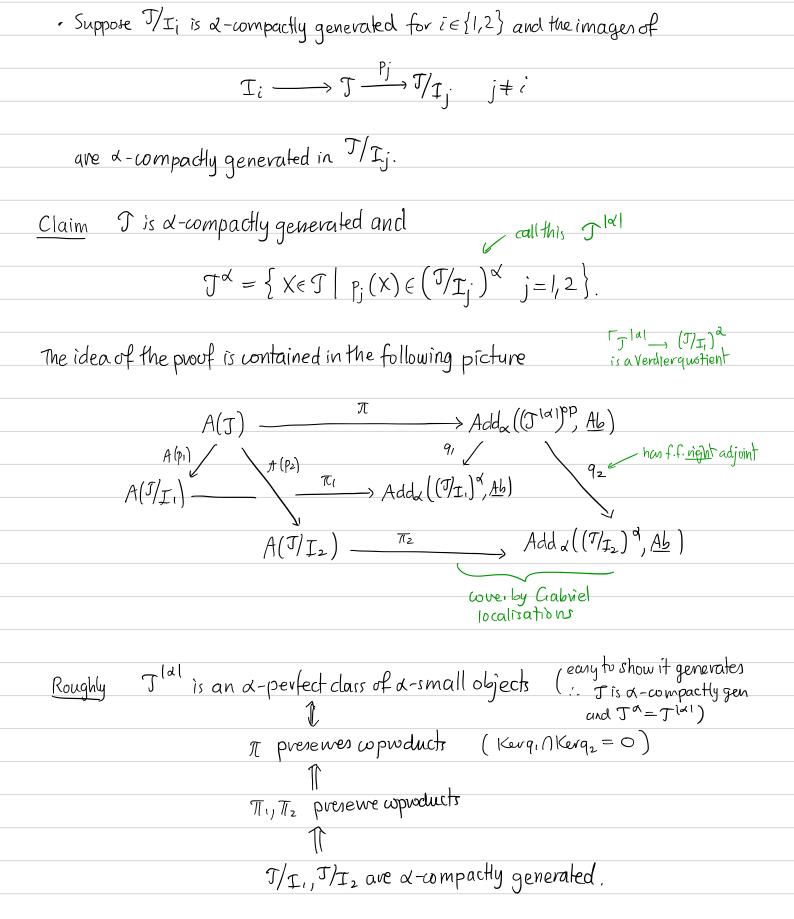
## Def T is a - compactly generated if

• 
$$\langle J^{\alpha} \rangle = J$$
 (so  $T \longrightarrow Add_{\alpha}((J^{\alpha})^{op}, \underline{Ab})$  sends nonzero  
objects to nonzero objects)

IF J is a -compactly generated for some x, it is well-generated.









3 Analogy (algebraic cases)

	Abelian	Triangulated
Banic example	MoclR, Raning Mod R, Rsmall preadditive	D(R), Raving D(C), Csmall DG-category
Fancy example	Mod R $\xrightarrow{\pi}$ A, a localisation, i.e. $\pi$ has a fully faithful vight adjoint, $A = Mod R/B$ .	$D(\mathcal{C}) \longrightarrow \mathcal{T}$ a localisation, 1-e. There is localising $S \subseteq D(\mathcal{C}), S = \langle S_0 \rangle, S_0 \ a \text{ set, with}$ $\mathcal{T} = D(\mathcal{C})/S.$
Gabriel-Popescu	A Gwthenclieck abelian $\Rightarrow A$ is fancy	J well-generated and algebraic ⇒ J is fancy (Porta)
Filhations	$A  \text{Gwothendieck} \Rightarrow \\ A = U_a  A^d$	$Twell-generated  \Rightarrow T = U a T^{a}                                     $
Adjoints	$F: \mathcal{A} \longrightarrow \mathcal{B}$ colim pros. + $\mathcal{A}$ Gvothendiech $\implies$ F has a vightadjoint	F: J→ S triangulated and 11-pres, + J well-generated ⇒ F has a right adjoint.