From critical points to extended topological field theories

Daniel Murfet
7/4/2020

therisingsea.org
Outline

I. From dynamical systems to monoidal bicategories

II. Extended topological quantum field theories
Dynamical systems

A general non-linear dynamical system is given by a system of DEs

\[
\begin{align*}
\dot{x}_1 &= F_1(x_1, \ldots, x_n) \\
\dot{x}_2 &= F_2(x_1, \ldots, x_n) \\
\dot{x}_n &= F_n(x_1, \ldots, x_n)
\end{align*}
\]

\[\dot{x} = F(x)\]

\[F : \mathbb{R}^n \rightarrow \mathbb{R}^n\]

An important class of dynamical systems are those which are conservative, in the sense that there is a scalar potential \( f : U \rightarrow \mathbb{R} \) with \( U \subseteq \mathbb{R}^n \),

\[F = \nabla f\]

\[\{\text{fixed points of system}\} = \{\text{critical points of } f\}\]

\[\nabla f(\xi) = 0\]
Example: Consider the system

\[
\begin{align*}
\dot{x}_1 &= x_1, \\
\dot{x}_2 &= -x_2
\end{align*}
\]

\[
\begin{pmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{pmatrix} =
\begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}
\]

Solution trajectories look like \( \mathbf{x}(t) = (Ae^t, Be^{-t}) \) for any \( A, B \in \mathbb{R} \).

The scalar potential governing this system is

\[
\begin{align*}
f &= \frac{1}{2}x_1^2 - \frac{1}{2}x_2^2 \\
\nabla f &= (x_1, x_2) \\
H_f &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\end{align*}
\]
Dynamical systems

To understand the dynamics near an isolated critical point of $f$, we need to analyse the Hessian of $f$, i.e.

$$H_f := \left( \frac{\partial^2 f}{\partial x_i \partial x_j} \right)_{1 \leq i, j \leq n},$$

its eigenvectors and eigenvalues. Actually the right way to think of this data is as a symmetric bilinear form on the tangent space $T_{\mathbf{x}^*}U$ at a critical point $\mathbf{x}^* \in U$, i.e.

$$\left( T_{\mathbf{x}^*}U, \langle \cdot, \cdot \rangle \right) \text{ where } \langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \rangle = \left. \frac{\partial^2 f}{\partial x_i \partial x_j} \right|_{x = \mathbf{x}^*},$$

with $\mathbf{u} = \mathbf{x} - \mathbf{x}^*$,

$$\mathbf{u} = H_f\big|_{\mathbf{x}^*} \mathbf{u} + \text{quadratic terms in } \mathbf{u} \text{ involving higher derivatives of } f$$

linear system
Morse Lemma If $H_f \big|_{x^*}$ is invertible (i.e., the corresponding bilinear form is nondegenerate) for an isolated critical pt. $x^*$ then there is a coordinate neighborhood around $x^*$ where

$$f = x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_n^2$$

so that in those coordinates

$$H_f \big|_{x^*} = \begin{pmatrix} 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 \end{pmatrix}$$

Def. A critical point $x^*$ is nondegenerate if $H_f \big|_{x^*}$ is invertible.

$\therefore$ locally $u = H_f \big|_{x^*} u$ $u = x - x^*$
**Quadratic spaces**

**Definition** The category $Q$ of quadratic spaces over $\mathbb{IR}$ has

- objects are f.d. vector spaces equipped with a nondegenerate symmetric bilinear form.

- morphisms $Q(V, W) = \{ T : V \to W \text{ linear} \mid <Tu, Tv> = <u, v> \ \forall u, v \}$.

**Example**

- $X_{p, q} : (\mathbb{R}^p \oplus \mathbb{R}^q, \left( \begin{smallmatrix} I_p & 0 \\ 0 & -I_q \end{smallmatrix} \right))$ is a representative set of objects (Sylvester's law of inertia)

- $X_{1, 0} = (\mathbb{IR}, (1)) \xrightarrow{(b)} (\mathbb{IR} \oplus \mathbb{IR}, (1 \ 0 \ -I)) = X_{1, 1}$ is a morphism.

- $(T_{x^*} U, \langle \cdot, \cdot \rangle) \left\langle \frac{\partial^2 f}{\partial x_i \partial x_j}, \frac{\partial^2 f}{\partial x_i \partial x_j} \right\rangle = \left. \frac{\partial^2 f}{\partial x_i \partial x_j} \right|_{x = x^*}$ at a nondeg. critical pt. $x^*$.

**Lemma** $Q$ is a symmetric monoidal category under direct sum of v. spaces.
Clifford algebras

Associated to each quadratic space $V$ is an algebra $C(V)$, the Clifford algebra which is universal among $\mathbb{R}$-algebras $C$ (associative and unital) equipped with a linear map $\ell: V \to C$ satisfying

$$\ell(v)\ell(w) + \ell(w)\ell(v) = 2\langle v, w \rangle \cdot 1_C.$$  

(so e.g. $\ell(v)^2 = \langle v, v \rangle \cdot 1_C$)

This thing exists, is naturally $\mathbb{Z}_2$-graded, $V \hookrightarrow C(V)$ is injective and $C(V)$ is $2^{\dim(V)}$ dimensional.

Examples $C(X_{0,0}) \cong \mathbb{R}$, $C(X_{0,1}) \cong \mathbb{C}$, $C(X_{0,2}) \cong \mathbb{H}$.
Lemma \( C(\cdot) \) is a strong monoidal functor \( Q \to Alg_{\mathbb{R}}^{\mathbb{Z}_2} \), i.e. there are natural isomorphisms \( C(0) \cong \mathbb{R} \) and

\[
C(V \otimes W) \cong C(V) \otimes_{\mathbb{R}} C(W).
\]

really direct sum!

critical point \( x^* \) of \( f \) \( \sim \) quadratic space \( (T_{x^*}U, H_f|_{x^*}) \)

\( \sim \) Clifford algebra \( C(T_{x^*}U, H_f|_{x^*}) \)

\( \sim \) Abelian category \( \text{Mod}_{\mathbb{Z}_2}^{\mathbb{Z}_2} C(T_{x^*}U, H_f|_{x^*}) \)

finite-dimensional \( \mathbb{Z}_2 \)-graded modules
Def: Nondegenerate isolated critical points form a bicategory $\text{Crit}^{\text{ndg}}_R$

- **objects** quadratic spaces $\mathcal{V}$
- $1$-**morphisms** $\mathcal{V} \to \mathcal{W}$ are $\mathbb{Z}_2$-graded finite-dimensional $C(\mathcal{W}) - C(\mathcal{V})$-bimodules.
- $2$-**morphisms** are bimodule homomorphisms.

**Proposition** $\text{Crit}^{\text{ndg}}_R$ is a symmetric monoidal bicategory in which every object is fully dualisable. (duals for objects and $1$-morphisms)

**Example**

- $\text{Crit}^{\text{ndg}}_R(0, \mathcal{V}) = \text{Mod}^{\mathbb{Z}_2} C(\mathcal{V})$. ($0 = X_{0,0} = I$)
- $X_{0,1}^\otimes \cong I$ (Bott periodicity)
Def. A bicategory $\mathcal{B}$ consists of

- a class of objects $a, b, c, \ldots$

- for each pair $a, b$ of objects a category $\mathcal{B}(a, b)$, objects of which are called 1-morphisms and denoted $X : a \rightarrow b$, and morphisms of which are called 2-morphisms.

- a composition functor for objects $a, b, c$

$$\beta(b, c) \times \beta(a, b) \longrightarrow \beta(a, c).$$

$$(y : b \rightarrow c, x : a \rightarrow b) \longmapsto (y \circ x : a \rightarrow c)$$

- unit 1-morphisms $1_a : a \rightarrow a$ for each object $a$

- 2-isomorphisms "unitors", "associators"

satisfying some coherence conditions (same as for monoidal categories)
**Def.** Let \( \mathcal{B}, \mathcal{C} \) be bicategories. A 2-functor \( F : \mathcal{B} \rightarrow \mathcal{C} \) is

- a function on objects \( a \mapsto F(a) \)
- functors \( \mathcal{B}(a, b) \rightarrow \mathcal{C}(F_a, F_b) \)
- natural isomorphisms

\[
F(Y) \circ F(X) \cong F(Y \circ X)
\]

\[
1_{F_a} \cong F(1_a)
\]

making some coherence diagrams commute.

**Example.** If \( \mathcal{B} \) is a bicategory, \( \mathcal{B}(a, -) : \mathcal{B} \rightarrow \mathsf{Cat} \) is a 2-functor, where \( \mathsf{Cat} \) denotes small categories, functors and natural transformations.
**Def** Let \( \mathcal{B}, \mathcal{C} \) be bicategories, \( F, G : \mathcal{B} \to \mathcal{C} \) 2-functors. A **pseudonatural transformation** \( \gamma : F \to G \) is

- a family of 1-morphisms \( \{ \gamma_a : Fa \to Ga \}_{a \in \text{ob}(\mathcal{B})} \)

- for each \( X : a \to b \) in \( \mathcal{B} \) a 2-isomorphism

\[
\begin{align*}
F_a & \xrightarrow{F_X} F_b \\
\gamma_a & \xRightarrow{\gamma_X} \gamma_b \\
G_a & \xrightarrow{G_X} G_b
\end{align*}
\]

subject to coherence conditions.

**Def** (notation as above) Given pseudonatural transformations \( \gamma, \psi : F \to G \), a **modification** \( \varrho : \gamma \to \psi \) is a family of 2-morphisms \( \{ \varrho_a : \gamma_a \to \psi_a \}_{a} \) satisfying a condition (omitted).
Lemma Let $\mathcal{B}, \mathcal{C}$ be bicategories, with $\mathcal{B}$ small. Then there is a bicategory $\text{Bicat}(\mathcal{B}, \mathcal{C})$.

- **objects** 2-functors

$$\text{Bicat}(\mathcal{B}, \mathcal{C})$$

- **1-morphisms** pseudonatural transformations

- **2-morphisms** modifications

**Monoidal bicategory (rough version)** is a bicategory $\mathcal{B}$ with

- tensor for objects $(a, b) \mapsto a \Box b$

- tensor for 1- and 2-morphisms, via a functor

$$\mathcal{B}(a_1, a_2) \times \mathcal{B}(b_1, b_2) \longrightarrow \mathcal{B}(a_1 \Box b_1, a_2 \Box b_2)$$

- associators, unitors, coherence.
**Def.** A monoidal bicategory is a bicategory \( \mathcal{B} \) equipped with

- a 2-functor \( \Box : \mathcal{B} \times \mathcal{B} \to \mathcal{B} \)

- an adjoint equivalence \( \alpha \) in \( \text{Bicat}(\mathcal{B} \times \mathcal{B} \times \mathcal{B}, \mathcal{B}) \) between the two legs of the following diagram (the associator)

\[
\begin{array}{ccc}
(\mathcal{B} \times \mathcal{B}) \times \mathcal{B} & \xrightarrow{\text{rebracket}} & \mathcal{B} \times (\mathcal{B} \times \mathcal{B}) \\
\Box \times 1 & \downarrow & \alpha \\
\mathcal{B} \times \mathcal{B} & \xrightarrow{\Rightarrow} & \mathcal{B} \times \mathcal{B} \\
1 \times \Box & \downarrow \\
\mathcal{B} \times \mathcal{B} & \xrightarrow{\Rightarrow} & \mathcal{B} \times \mathcal{B}
\end{array}
\]

i.e. \( \alpha \) is a pseudonatural transformation.
an invertible modification $\pi$, the pentagonator

$$
\begin{align*}
(A \circ B) \circ (C \circ D) \\
\alpha \\
((A \circ B) \circ C) \circ D \\
\alpha \\
(A \circ (B \circ C)) \circ D \\
\alpha \\
A \circ ((B \circ C) \circ D) \\
\alpha
\end{align*}
$$

+ units, unitors and lots of coherence!
Def: A symmetric monoidal bicategory is a monoidal bicategory $\mathcal{B}$ with

- an adjoint equivalence $\beta$ in $\text{Bicat}(\mathcal{B} \times \mathcal{B}, \mathcal{B})$ between the legs of

\[
\begin{array}{ccc}
\mathcal{B} \times \mathcal{B} & \xrightarrow{\square} & \mathcal{B} \\
\downarrow \beta & & \downarrow \beta \\
\mathcal{B} \times \mathcal{B} & \xrightarrow{\text{swap}} & \mathcal{B} \times \mathcal{B}
\end{array}
\]

i.e., $a \Box b$ and $b \Box a$

- an invertible modification called syllepsis

\[
\begin{array}{ccc}
\mathcal{B} & \xrightarrow{1_{a \Box b}} & \mathcal{B} \\
\downarrow \beta & & \downarrow \beta \\
\mathcal{B} & \xrightarrow{b \Box a} & \mathcal{B}
\end{array}
\]

- invertible modifications relating $\beta$ and the associator + coherence
Examples

1. \((\text{Cat}, \times)\) categories, functors, natural transformations, Cartesian product

2. \((\text{Alg}_k, \otimes_k)\) algebras, bimodules, bimodule maps, tensor

3. \((\text{Crit}^{ndg}_{\mathbb{R}}, \oplus)\) quadratic spaces, Clifford bimodules and maps, direct sum.

References (not a historical survey!)

- Chris Schommer-Pries' PhD thesis
- Nick Gurski “Loop spaces, and coherence for monoidal and braided monoidal bicategories”.
Duals in symmetric monoidal bicategories

Let $\mathcal{B}$ be a monoidal bicategory. A right dual to an object $a$ is $a^*$ and 1-morphisms $\text{ev}_a : a \otimes a^* \to 1$, $\text{coev}_a : 1 \to a^* \otimes a$

and cosp isomorphisms in $\mathcal{B}(a,a)$

$$
\begin{align*}
& a \xrightarrow{1 \otimes \text{coev}_a} a \otimes 1 \xrightarrow{a \otimes (a^* \otimes a)} (a \otimes a^*) \otimes a \xrightarrow{\text{ev}_a \otimes 1} 1 \otimes a \xrightarrow{\text{Id}} a \\
\end{align*}
$$

$$
\begin{align*}
& a^* \xrightarrow{1 \otimes a^*} 1 \otimes a^* \xrightarrow{(a^* \otimes a) \otimes a^*} a^* \otimes (a \otimes a^*) \xrightarrow{1 \otimes \text{ev}_a} a^* \otimes 1 \xrightarrow{\text{Id}} a^* \\
\end{align*}
$$

Lemma In a symmetric monoidal category every right dual is also a left dual.
Duals in symmetric monoidal bicategories

**Def.** Let $\mathcal{B}$ be a symmetric monoidal bicategory. An object $a$ is **fully dualisable** if it has a dual object such that both $\text{eva}_a$ and $\text{coeva}_a$ have both left and right adjoints.

- Every object in $\text{Crit}^{\text{ndg}}$ is fully dualisable $(V, B)^\ast := (V, -B)$.

**Theorem (Pstragowski)** [2D cobordism hypothesis] There is an equivalence

$$\text{Bicat}_{\text{sym, mon}}(\text{Bord}_2^f, \mathcal{B}) \cong K(\mathcal{B}^\text{fd})$$

- framed bordism bicategory
- fully dualisable objects
- core, i.e. keep equivalences and 2-isomorphisms
\[ f = \frac{1}{2} x_1^2 - \frac{1}{2} x_2^2 \]

\[ \text{Phase portrait} \]

\[ \text{Crit}^{\text{ndg}}_{\mathbb{R}} \]

\[ \bullet (T_{x^*} U, H_f \mid_{x^*}) \]
\[ f = \frac{1}{2} x_1^2 - \frac{1}{2} x_2^2 \]

Around an isolated (degenerate) critical point \( x^* \)

\[ \dot{u} = H_f \big|_{x^*} u + \text{quadratic terms in } u \]

where \( u = x - x^* \), the dynamics do depend on the higher derivatives of the potential \( f \).
\[ f = \frac{1}{2} x_1^2 - \frac{1}{2} x_2^2 \]

**Phase portrait**

\[ \mathbf{u} = \left. H_f \right|_{x^*} \mathbf{u} + \text{quadratic terms in } \mathbf{u} \]

involving higher derivatives of the potential \( f \).

**Question**: What algebra to associate to \( (f, x^*) \)?

- **linear system**

  * reduce to \( C(T_{x^*} U, H_f \mid_{x^*}) \) in the nondeg. case
  
  * form a symmetric monoidal bicategory
\[ f = \frac{1}{2} x_1^2 - \frac{1}{2} x_2^2 \]

\[ f = \frac{1}{3} x_1^3 - \frac{1}{2} x_2^2 \]

**Phase portrait**

- nondegenerate critical point
- degenerate critical point

**Question** What algebra to associate to \((f, x^*)\)?

- reduce to \(C(T_x^* U, Hf |_{x^*})\) in the nondeg. case
- form a symmetric monoidal bicategory
Question: What algebra to associate to \((f, x^*)\) ?

- reduce to \(C(T_x^* U, Hf|_{x^*})\) in the nondeg. case
- form a symmetric monoidal bicategory
Matrix factorisations

Let $X$ be a $\mathbb{Z}_2$-graded f.d. module over the Clifford algebra

\[ C(X_{p,q}) : \text{ generated by } \sigma_1, \ldots, \sigma_{p+q} \text{ subject to } \]

\[
\sigma_i^2 = \ldots = \sigma_p^2 = 1 \\
\sigma_{p+1}^2 = \ldots = \sigma_{p+q}^2 = -1 \\
\sigma_i \sigma_j + \sigma_j \sigma_i = 0 \quad i \neq j
\]
Matrix factorisations

Let $X$ be a $\mathbb{Z}_2$-graded f.d. module over the Clifford algebra

$\mathcal{C}(X_{p,q})$: generated by $\sigma_1, \ldots, \sigma_{p+q}$ subject to

- $\sigma_1^2 = \ldots = \sigma_p^2 = 1$
- $\sigma_{p+1}^2 = \ldots = \sigma_{p+q}^2 = -1$
- $\sigma_i \sigma_j + \sigma_j \sigma_i = 0 \quad i \neq j$

Set $A = \mathbb{R}[x_1, \ldots, x_{p+q}]$, and

$\mathcal{C}(X \otimes_A A) \supset \mathfrak{g} = \sum_{i=1}^{n} x_i \sigma_i$

$\mathcal{O}_2$-graded free $A$-module

acting on $X \otimes_A A$
Potentials

Let \( k \) be a commutative \( \mathbb{Q} \)-algebra, then \( f \in \mathbb{R} = k[x_1, \ldots, x_n] \) is called a potential if

(i) \( \partial x_i f, \ldots, \partial x_n f \) is quasi-regular

(ii) \( \mathbb{R} / (\partial x_i f, \ldots, \partial x_n f) \) is a f.g. free \( k \)-module

(iii) the Koszul complex of \( \partial x_i f, \ldots, \partial x_n f \) is exact outside \( \operatorname{deg} \mathbb{O} \).

Example \( f \in \mathbb{C}[x_1, \ldots, x_n] \) such that \( \operatorname{dim}_\mathbb{C} \mathbb{C}[x_1, \ldots, x_n] / (\partial x_i f, \ldots, \partial x_n f) < \infty \).

(isolated critical points)

Defn. The DA-category \( \mathcal{A} = \operatorname{mf}(\mathbb{R}, f) \) has

- objects f. rank matrix factorisations of \( f \), i.e. \( X \in \mathbb{R} \) \( d^2_x = f \cdot 1_x \).

- morphisms \( \mathcal{A}(X, Y) = (\operatorname{Hom}_\mathbb{R}(X, Y), \alpha \mapsto d_Y \alpha - (-1)^{1\alpha} \alpha d_X) \).

This is a \( \mathbb{Z}_2 \)-graded DA-category over \( \mathbb{R} \).
Remarks

• \( \text{hmf}(R, f) := H^0 \text{mf}(R, f) \) is triangulated (Calabi-Yau).

• Given a quadratic space \( V \) with associated quadratic \( f \in \text{Sym}(V^*) \):

\[
\text{Mod}_{\mathbb{Z}^2} \mathcal{C}(V) \cong \text{hmf}(\text{Sym}(V^*), f)^\infty
\]

(Buchweitz-Eisenbud-Herzog)
Remains • hmf(R, f) := \text{H}^0 \text{mf}(R, f) is triangulated (Calabi-Yau).

• Given a quadratic space \( V \) with associated quadratic \( f \in \text{Sym}(V^*) \)

\[
\text{Mod}_{f,a}^\mathbb{Z} \mathcal{C}(V) \cong \text{hmf}(\text{Sym}(V^*), f)^\omega
\]

(Buchweitz–Eisenbud-Herzog)

From a potential \( f \) to an \( A_\infty \)-algebra \( A_f \)

Assume \( k \) is a field and \( \text{Sing}(f) = \{ 0 \} \). Then there is a standard generator

\[
\text{thick}(G) = \text{hmf}(R, f)^\omega
\]

\[
\text{perf End}_R(G) \cong \text{hmf}(R, f)^\omega \quad (\text{Keller–Lefevre})
\]

\[
\text{perf}_\infty H^* \text{End}_R(G) \cong \text{hmf}(R, f)^\omega
\]

\( A_\infty \)-algebra \( A_f \), is a Clifford algebra for quadratic \( f \).

\( A_\infty \)-products package higher derivatives of \( f \).
Pseudo-def $\text{Crit}_R$ is the bicategory of $A_\infty$-algebras $A(f, z^*)$ associated to isolated critical points, $A_\infty$-bimodules and $A_\infty$-bimodule maps.

**Theorem** (Carqueville-Montoya '18) $\text{Crit}_R$ is a symmetric monoidal bicategory in which every object is fully dualisable, and therefore determines an extended 2D framed TFT

$$\text{Bord}_2^\text{fr} \longrightarrow \text{Crit}_R.$$ 

Moreover $\text{Crit}^\text{ndg}_R \subset \text{Crit}_R$. 

$\uparrow$ essentially due to Buchweitz-Eisenbud-Herzog.
Sketch of $\mathcal{LG}_k$ (e.g. Criteria) (bicategory of Landau–Ginzburg models)

$k$ any commutative ring

\[ (y, V(y)) \quad W \square V = W + V \]

\[ 1 = (\phi, 0) \]

\[ W^* = -W \]

\[ G \otimes_{R[y]} E \]

Reference: N. Carqueville, DM. "Adjoints and defects in Landau–Ginzburg models"
Def \[ \text{Let } X^k \text{ be a manifold, possibly with corners. If } k \leq 2 \text{ a 2-halo over } X \text{ is a sequence of inclusions of pro-manifolds} \]
\[
X \leq \hat{X}_1 \leq \hat{X}_2
\]
such that \( X \leq \hat{X}_1, X \leq \hat{X}_2 \) have the structure of cooriented halations of \( \text{dim } 1, 2 \) respectively.

Diagram:
- \( X \)
- \( \hat{X}_1 \) labeled "germ of 1-manifold"
- \( \hat{X}_2 \) labeled "germ of 2-manifold"
Theorem (Schommer-Pries) There is a symmetric monoidal bicategory $\text{Bord}_2^{fr}$

Objects: framed 2-haloed 0-manifolds

1-morphisms: framed 2-haloed 1-bordisms

$\partial \omega \cong \partial_{in} \omega \sqcup \partial_{out} \omega$

2-morphisms: framed 2-haloed 2-bordisms / $\simeq$

$\partial \alpha = \partial_0 \cup \partial_\omega$ $\partial_0 \cup \partial_\omega = \partial_0$

$\partial \omega \cong \partial_0 \sqcup \partial_\omega$ $\partial_\omega \sqcup \partial_0 = 2\partial_0$

$\partial_0 \cup \partial_\omega = A \times I \sqcup B \times I$
Structure of $L^G_k$ under control $\implies$ one can actually compute this TQFT

$$\text{Bicat}_{sym\cdot mon}(\text{Bord}^k_{fr}, L^G_k) \cong K(L^G_k)$$

**Application** The “TQFT with corners” constructed by Khovanov and Rozansky can be derived/corrected using the cobordism hypothesis as extended TQFTs

$$\text{pt}^+ \mapsto x^{N+1} \in L^G_R.$$  

Proving this uses explicit formulas for ev, coev in $L^G_k$.  

Proving this uses explicit formulas for ev, coev in $L^G_k$. 

\[ 1 \square 1 = 1 \]

\[ y^{N+1} \square (x^{N+1})^* \]

\[ y^{N+1} \square (x^{N+1})^* \]

\[ y^{N+1} - x^{N+1} \]

\[ \text{coev} \circ \text{ev} \]

\[ y^{N+1} - x^{N+1} \]

\[ p.89 \text{ Montoya's thesis} \]