Lecture 3: An introduction to Cobordisms

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Definition 1 (Topological manifold)**.** An n-dimensional topological manifold with boundary, M, is a Hausdorff topological space such that:

- $\forall x \in M^{\circ}$, there is an open set U such that $x \in U \cong B_{\varepsilon}(0) \subset \mathbb{R}^n$.
- If $x \in \partial M$, then there is an open set U such that $x \in U \cong B_{\varepsilon}(0) \cap \{x \in \mathbb{R}^n | x_n \geq 0\}$

Definition 2. Cobordism [1] Given any two compact $(n-1)$ dimensional manifolds Σ_1 and Σ_2, a cobordism between them is a compact n dimensional manifold M such that $\partial M = \Sigma_1 \sqcup \Sigma_2$

Remark 3 (Etymology)*.* A single manifold Σ which formed the border of some manifold M was called *bordant.* When two manifolds together form the boundary of some M, they are called *cobordant.*

Some examples of cobordisms in one and two dimensions are given in figures and . Note that all one dimensional cobordisms are homeomorphic to the disjoint union of some number of intervals, between the disjoint unions of some number of points. Similarly, all cobordant one dimensional manifolds are homeomorphic to the disjoint union of some number of circles.

Remark 4*.* Note that cobordisms are not embedded in any space. Therefore the two images in figure represent the same cobordism.

Definition 5 (Alternating)**.** Let V be a vector space. Then a linear map $\beta : V \otimes V \to k$ is alternating if $\beta(a, a) = 0$ for all $a \in V$

Lemma 6. *Let $\beta : V \otimes V \to \mathbb{R}$ be an alternating linear map. Then $\beta(a, b) = -\beta(b, a)$*

Proof.

$$\begin{aligned} 0 &= \beta(a + b, a + b) \\ &= \beta(a, a) + \beta(a, b) + \beta(b, a) + \beta(bb) \\ &= \beta(a, b) + \beta(b, a) \end{aligned}$$

$\implies -\beta(b, a) = \beta(a, b)$ □

In particular, call (u, v) positive if $\nu(u \otimes v) > 0$, and negative if $\nu(u \otimes v) < 0$.

Given an oriented manifold M with boundary, we can induce an orientation on ∂M.

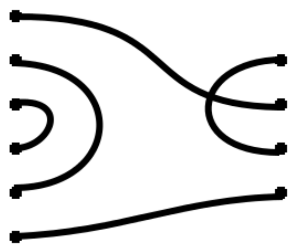


Figure 1: An example of a 1d cobordism from $\{1, 2, 3, 4, 5, 6\}$ to $\{1, 2, 3, 4\}$

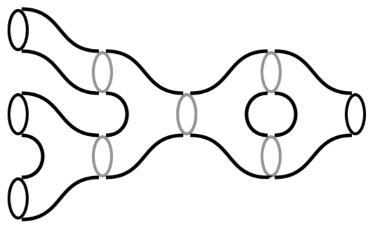


Figure 2: An example of a 2d cobordism from $S^1 \sqcup S^1 \sqcup S^1$ to S^1

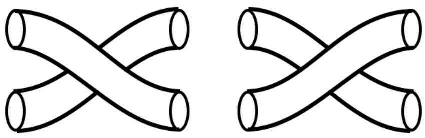
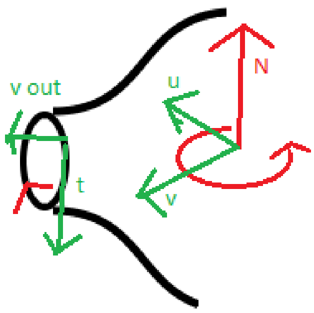


Figure 3: These two images represent the same cobordism, since cobordisms are not embedded in any space



Let $x \in M$. Consider the dot product $\nu : T_xM \otimes T_xM \to \mathbb{R}$, which is an alternating linear map. In the diagram, intuitively u comes before v and so $\nu(u, v)$ is positive, but $\nu(v, u)$ is negative. Now to induce an orientation on the boundary, $\Sigma = \partial M$, let $x \in \partial M$. Note that there is a map $T_x\Sigma \to T_xM$ induced by the embedding of Σ onto ∂M. There is a particular choice of vector v_{out} which can be made. For any $t \in T_xM$, define a map $\nu_\sigma : T_x\Sigma \to \mathbb{R} : t \mapsto \nu(v_{out}$. This induces an orientation on the boundary.

Definition 7 (Oriented cobordism)**.** Given any two compact, oriented, $(n-1)$ dimensional manifolds Σ_1 and Σ_2, an oriented cobordism $M : \Sigma_1 \to \Sigma_2$ is a compact, oriented, n dimensional manifold along with smooth orientation preserving embeddings $f_1 : -\Sigma_1 \to \partial M$ and $f_2 : \Sigma_2 \to \partial M$, (where $-\Sigma$ is Σ with the orientation reversed) such that $f_1 \sqcup f_2 : -\Sigma_1 \sqcup \Sigma_2 \to \partial M$ is a diffeomorphism.

Example 8 (1D oriented cobordisms)**.** Let p be a point with orientation $+$, and consider the interval $[0, 1]$ with orientation as in figure . This induces a negative orientation at 0 and a positive orientation at 1. This allows $[0, 1]$ to be an oriented cobordism from p to p, with the maps $f_1 : -p \to 0 \subset \partial[0, 1]$ and $f_2 : p \to 1 \subset \partial[0, 1]$, which are clearly orientation preserving.

Now let q be a point with orientation $-$. Then there are orientation preserving maps $f_1 : -(p \sqcup q) \to \{0, 1\} \subset \partial[0, 1]$ and $f_2 : \emptyset \to \partial[0, 1]$. These two different options are shown in figure .

Example 9 (2D oriented cobordisms)**.** Consider the manifold depicted in figure , a 2d manifold with three boundary circles. Let $\Sigma_1 = S^1 \sqcup S^2$ and $\Sigma_2 = S^1$. Then we can have two maps, one $-\Sigma_1 \to \partial M$ on the left of

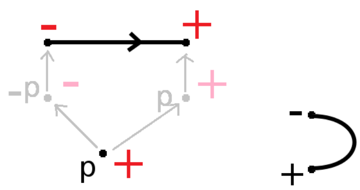


Figure 4: On the left is an example of a cobordism from p to p, while on the right is an example of a cobordism from $p \sqcup q$ to $\emptyset$

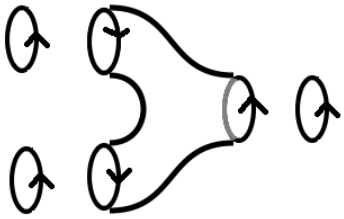


Figure 5: In this example we have a cobordism from two circles to one, with each boundary point having an induced orientation. Each circle on the left is oriented opposite to the boundary, so they are part of Σ_1, and the circle on the right is oriented the same way as the boundary, so it is Σ_2

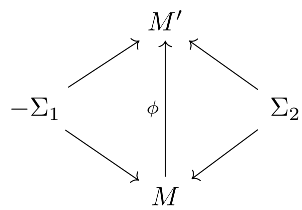
Figure 6: Only the top left circle is oriented opposite to the boundary, so $\Sigma_1 = S^1$. The right and bottom left circles are oriented the same way as the boundary, so they must form Σ_2 for this to be a cobordism. Thus, this is a cobordism from one circle to two, and if following the convention of drawing cobordisms from left to right it would more properly be draw as the image on the right

the image, and the other $\Sigma_2 \to \partial M$ on the right, which satisfy the conditions for M to be an oriented cobordism from Σ_1 to Σ_2

Now consider what happens in the orientation of one of the circles on the left is swapped, as in figure . This circle cannot be part of Σ_1 since we wish to preserve orientations rather than flip them. Therefore, let $\Sigma_1 = S^1$ and $\Sigma_2 = S^1 \sqcup S^1$. Conventionally, this cobordism would be drawn so that Σ_1, or the "in" boundary, is on the left, and Σ_2, or the "out" boundary, is on the right.

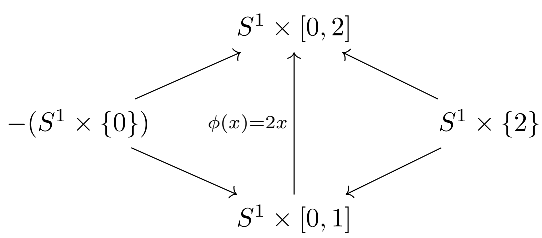
It seems that these two depictions in figure should represent the same cobordism. In fact, most of the time, cobordism classes will be used rather than cobordisms, and for this we require some notion of *equivalence*.

Definition 10 (Equivalence of cobordisms)**.** Let Σ_1 and Σ_2 be $n-1$ dimensional manifolds such that M and M' are oriented cobordisms between them. Then M and M' are equivalent if there exists some diffeomorphism ϕ such that the following diagram commutes:



Example 11. The cylinder $\Sigma \times [0,1]$ is equivalent to the cylinder $\Sigma \times [0,2]$

Proof. Given the following diagram, all that is required is to check that it commutes.



□

Thus, from now on we can talk about cobordism classes, rather than cobordisms themselves Eventually, we will form a category whose objects are $n-1$ dimensional manifolds, with cobordisms as morphisms. Thus, we require some notion of gluing two cobordisms together.

Definition 12. Let $f_1 : \Sigma \hookrightarrow M_1$ and $f_2 : \Sigma \hookrightarrow M_2$ be injective maps between topological spaces. Now $M_1 \sqcup_\Sigma M_2$ is defined to be the disjoint union of M_1 and M_2 quotiented out by the following equivalence relation. Let $m_1 \in M_1 = m_2 \in M_2$ iff there is an $x \in \Sigma$ such that $f_1(x) = m_1$ and $f_2(x) = m_2$. We also require that $x \equiv x$.

Thus,

$$M_1 \longrightarrow M_1 \sqcup_\Sigma M_2 \longleftarrow M_2$$

A subset $u \in M_1 \sqcup_\Sigma M_2$ is defined to be open if both its inverse images are open.

Lemma 13. *The above equivalence relation is indeed an equivalence relation*

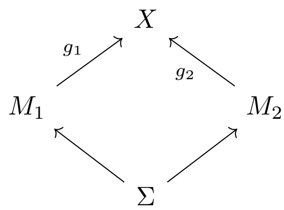
Proof. Firstly, due to the extra condition $x \equiv x$, this relation is reflexive. Then, if $m_1 \equiv m_2$, then we can swap around M_1 and M_2 to arrive at $m_2 \equiv m_1$ so the relation is symmetric.

The only difficult part is transitivity. For this, there are multiple cases to consider. Let $m_1 \equiv m_2$ and $m_2 \equiv m_3$.

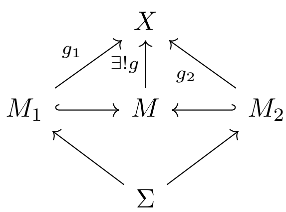
- If $m_1, m_2 \in M_1$ then $m_1 = m_2$ and thus $m_1 \equiv m_3$. Similarly for the case that m_2 and m_3 lie on the same manifold.
- If all three lie on the same manifold, say M_1, then $m_1 = m_2 = m_3$.
- If $m_1, m_3 \in M_1$ and $m_2 \in M_2$, then there is an $x \in \Sigma$ such that $f_1(x) = m_1$ and $f_2(x) = m_2$, and a $y \in \Sigma$ such that $f_1(y) = m_3$ and $f_2(y) = m_2$. Thus we have that $f_2(x) = f_2(y)$. Since f_2 is injective, $x = y$ and so $f_1(x) = f_1(y) \implies m_1 = m_3$ and so $m_1 \equiv m_3$.

□

Lemma 14. *For every commutative diagram*



There exists a unique continuous map g such that

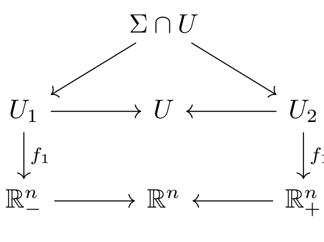


Note that this is actually a pushout.

In order to glue topological manifolds together in this manner, it must be shown that $M = M_1 \sqcup_\Sigma M_2$ is a topological manifold. To do this, an atlas must be constructed on M.

Lemma 15. *$M_1 \sqcup_\Sigma M_2$ is a topological manifold*

Proof. In order to show that this is a manifold, we require charts. All points which are not on Σ already have charts, so it remains to find charts for those points on Σ. Let x be a point on Σ, and $U \subset M_1 \sqcup_\Sigma M_2$ an open subset containing x. Note that $U_1 := U \cap M_1$ and $U_2 := U \cap M_2$ are open, and that $U = U_1 \sqcup_{\Sigma \cap U} U_2$. Let U be such that U_1 and U_2 are charts with functions $f_1 : U_1 \to \mathbb{R}^n_-$ and $f_2 : U_2 \to \mathbb{R}^n_+$. Now there is a diagram:



By lemma 14 there exists some $f : U \to \mathbb{R}^n$. □

Note that f_1 is an isomorphism $U_1 \cong B_\varepsilon(0) \cap \mathbb{R}^n_-$. Same for f_2. These induce f an isomorphism $U \cong B_\varepsilon(0)$

Example 16 (Gluing two cylinders)**.** Let Σ_0, Σ_1 and Σ_3 be manifolds equivalent to the circle. Then let M_1 and M_2 be cobordisms from Σ_0 to Σ_1 and Σ_1 to Σ_2, respectively. These two manifolds M_1 and M_2 are both equivalent to cylinders, with $\phi_1 : M_1 \to \Sigma_1 \times [0, 1]$ and $\phi_2 : M_2 \to \Sigma_1 \times [1, 2]$.

Let $S := \Sigma_1 \times [0, 2]$, with ψ being the diffeomorphism from $\Sigma_1 \sqcup \Sigma_2$ to S

$$\phi := \phi_1 \sqcup_{\Sigma_1} \phi_2 : M_1 \sqcup_{\Sigma_1} M_2 \to S$$

is a diffeomorphism.

References

[1] Joachim Kock. *Frobenius Algebras and 2D Topological Quantum Field Theories.*