From critical points to $A_\infty$-categories

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Outline

I. From dynamical systems to $A_\infty$-products

II. Constructing $A_\infty$-categories via idempotents
Dynamical systems

A general non-linear dynamical system is given by a system of DEs

\[
\begin{align*}
    \dot{x}_1 &= F_1(x_1, \ldots, x_n) \\
    \dot{x}_2 &= F_2(x_1, \ldots, x_n) \\
    \vdots \quad & \quad \vdots \\
    \dot{x}_n &= F_n(x_1, \ldots, x_n)
\end{align*}
\]

\[
\dot{\mathbf{x}} = F(\mathbf{x})
\]

\[F : \mathbb{R}^n \rightarrow \mathbb{R}^n\]

An important class of dynamical systems are those which are conservative, in the sense that there is a scalar potential \( f : U \rightarrow \mathbb{R} \) with \( U \subseteq \mathbb{R}^n \),

\[
F = \nabla f.
\]

\[
\{ \text{fixed points of system} \} = \{ \text{critical points of } f \} \quad \nabla f(\mathbf{x}) = 0
\]
**Example** Consider the system

\[
\begin{align*}
\dot{x}_1 &= x_1 \\
\dot{x}_2 &= -x_2
\end{align*}
\]

Solution trajectories look like \( x(t) = (Ae^t, Be^{-t}) \) for any \( A, B \in \mathbb{R} \).

The scalar potential governing this system is

\[
\begin{align*}
f &= \frac{1}{2}x_1^2 - \frac{1}{2}x_2^2 \\
\nabla f &= \begin{pmatrix} x_1 & 0 \\ 0 & -x_2 \end{pmatrix} \\
H_f &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\end{align*}
\]
Dynamical systems

To understand the dynamics near an isolated critical point of \( f \) we need to analyse the Hessian of \( f \), i.e.

\[
H_f := \left( \frac{\partial^2 f}{\partial x_i \partial x_j} \right)_{1 \leq i, j \leq n},
\]

its eigenvectors and eigenvalues. Actually the right way to think of this data is as a symmetric bilinear form on the tangent space \( T_{x^*} U \) at a critical point \( x^* \in U \), i.e.

\[
(T_{x^*} U, \langle , \rangle) \quad \text{where} \quad \langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \rangle = \left. \frac{\partial^2 f}{\partial x_i \partial x_j} \right|_{x = x^*}
\]

with \( u = x - x^* \),

\[
\dot{u} = H_f \big|_{x^*} u + \text{quadratic terms in } u \text{ involving higher derivatives of } f
\]

linear system
Morse Lemma If \( H_f \big|_{x^*} \) is invertible (i.e., the corresponding bilinear form is nondegenerate) for an isolated critical pt. \( x^* \) then there is a coordinate neighborhood around \( x^* \) where

\[
 f = x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_n^2
\]

so that in those coordinates

\[
 H_f \big|_{x^*} = 
\begin{pmatrix}
 1 & & & & \\
 & \ddots & & & \\
 & & 1 & & \\
 & & & -1 & \\
 & & & & -1
\end{pmatrix}
\]

Def. A critical point \( x^* \) is nondegenerate if \( H_f \big|_{x^*} \) is invertible.

\[
 \therefore \text{ locally } \quad u = H_f \big|_{x^*} u \quad \quad u = x - x^*
\]
**Quadratic spaces**

**Definition** The category $Q$ of quadratic spaces over $\mathbb{R}$ has

- objects are f.d. vector spaces equipped with a nondegenerate symmetric bilinear form.

- morphisms $Q(V,W) = \{ T: V \to W \text{ linear} \mid \langle Tu, Tv \rangle = \langle u, v \rangle \ \forall u, v \}$. 

**Example**

- $X_{p,q} := (\mathbb{R}^p \oplus \mathbb{R}^q, \left( \begin{smallmatrix} I_p & 0 \\ 0 & -I_q \end{smallmatrix} \right))$ is a representative set of objects (Sylvester's law of inertia)

- $X_{1,0} = (\mathbb{R}, (1)) \xrightarrow{(1)} (\mathbb{R} \oplus \mathbb{R}, \left( \begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} \right)) = X_{1,1}$ is a morphism.

- $(T^* U, \langle , \rangle) \quad \left\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right\rangle = \frac{\partial^2 f}{\partial x_i \partial x_j} \bigg|_{x = x^*}$ at a nondeg. critical pt. $x^*$.

**Lemma** $Q$ is a symmetric monoidal category under direct sum of $v$-spaces.
Clifford algebras

Associated to each quadratic space $V$ is an algebra $C(V)$, the Clifford algebra which is universal among $\mathbb{R}$-algebras $C$ (associative and unital) equipped with a linear map $l : V \to C$ satisfying

$$l(v)l(w) + l(w)l(v) = 2\langle v, w \rangle \cdot 1_C.$$ (so e.g. $l(v)^2 = \langle v, v \rangle \cdot 1_C$)

This thing exists, is naturally $\mathbb{Z}_2$-graded, $V \hookrightarrow C(V)^2$ is injective and $C(V)$ is $2^{\dim(V)}$ dimensional.

Examples $C(X_{0,0}) \cong \mathbb{R}$, $C(X_{0,1}) \cong \mathbb{C}$, $C(X_{0,2}) \cong \mathbb{H}$
Lemma $C(-)$ is a strong monoidal functor $Q \rightarrow \text{Alg}_{\mathbb{R}}^{\mathbb{Z}_2}$, i.e. there are natural isomorphisms $C(0) \cong \mathbb{R}$ and

$$C(V \otimes W) \cong C(V) \otimes_{\mathbb{R}} C(W).$$

really direct sum!

Critical point $x^*$ of $f$ $\rightarrow$ quadratic space $(T_{x^*}U, Hf|_{x^*})$

$\rightarrow$ Clifford algebra $C(T_{x^*}U, Hf|_{x^*})$

$\rightarrow$ Abelian category $\text{Mod}^{\mathbb{Z}_2} C(T_{x^*}U, Hf|_{x^*})$

finite-dimensional $\mathbb{Z}_2$-graded modules
Def. Nondegenerate isolated critical points form a bicategory $\text{Crit}_{\mathcal{R}}^{\text{ndg}}$

- objects: quadratic spaces $\mathcal{V}$
- 1-morphisms $\mathcal{V} \to \mathcal{W}$ are $\mathbb{Z}_2$-graded finite-dimensional $\mathcal{C}(\mathcal{W}) - \mathcal{C}(\mathcal{V})$-bimodules.
- 2-morphisms are bimodule homomorphisms.

Proposition. $\text{Crit}_{\mathcal{R}}^{\text{ndg}}$ is a symmetric monoidal bicategory in which every object is fully dualisable. (duals for objects and 1-morphisms)

Example.
- $\text{Crit}_{\mathcal{R}}^{\text{ndg}}(O, \mathcal{V}) = \text{Mod}^{\mathbb{Z}_2} \mathcal{C}(\mathcal{V})$. ($O = X_{0,0} = \mathbb{I}$)
- $X_{0,1}^{\otimes 8} \simeq \mathbb{I}$ (Bott periodicity)
\[ f = \frac{1}{2} x_1^2 - \frac{1}{2} x_2^2 \]

Phase portrait

nondegenerate critical point

\[ \text{Crit}_{\mathbb{R}}^{\text{ndg}} \]

\[ \cdot \left( T_z^+ U, H_f \big|_{z^*} \right) \]
\[ f = \frac{1}{2} x_1^2 - \frac{1}{2} x_2^2 \]

Phase portrait

Around an isolated (degenerate) critical point \( \mathbf{x}^* \)

\[ \dot{\mathbf{u}} = H_f \big|_{\mathbf{x}^*} \mathbf{u} + \text{quadratic terms in } \mathbf{u} \]

Linear system

where \( \mathbf{u} = \mathbf{x} - \mathbf{x}^* \), the dynamics do depend on the higher derivatives of \( f \).
\[ f = \frac{1}{2} x_1^2 - \frac{1}{2} x_2^2 \]

Phase portrait

nondegenerate critical point

\[ \text{Crit}_{\mathbb{R}}^{\text{ndg}} \]

\[ \bullet (T_{x^*} U, H_f |_{x^*}) \]

\[ f = \frac{1}{3} x_1^3 - \frac{1}{2} x_2^2 \]

degenerate critical pt.

\[ \dot{x}_1 = x_1^2 \]
\[ \dot{x}_2 = -x_2 \]

\[ \dot{u} = H_f |_{x^*} u + \text{quadratic terms in } u \text{ involving higher derivatives of the potential } f. \]

Question: What algebra to associate to \((f, x^*)\)?

- reduce to \(C(T_{x^*} U, H_f |_{x^*})\) in the nondeg. case
- form a symmetric monoidal bicategory
\[ f = \frac{1}{2} x_1^2 - \frac{1}{2} x_L^2 \]

Phase portrait

\[ f = \frac{1}{3} x_1^3 - \frac{1}{2} x_2^2 \]

degenerate critical pt.

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\[ \dot{x}_1 = x_1^2 \]
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**Question:** What algebra to associate to \((f, x^*)\)?

- reduce to \(\mathcal{C}(T_{x^*} U, H_f |_{x^*})\) in the nondeg. case
- form a symmetric monoidal bicategory
Matrix factorisations

Let $X$ be a $\mathbb{Z}_2$-graded f.d. module over the Clifford algebra

$$C(X_{p,q}) : \text{ generated by } \sigma_1, \ldots, \sigma_{p+q} \text{ subject to}$$

$$\sigma_1^2 = \ldots = \sigma_p^2 = 1$$
$$\sigma_{p+1}^2 = \ldots = \sigma_{p+q}^2 = -1$$
$$\sigma_i \sigma_j + \sigma_j \sigma_i = 0 \quad i \neq j$$
Matrix factorisations

Let $X$ be a $\mathbb{Z}_2$-graded f.d. module over the Clifford algebra $C(\mathbb{R}[x_1, \ldots, x_{p+q}])$ generated by $\sigma_1, \ldots, \sigma_{p+q}$ subject to

\[
\begin{align*}
\sigma_1^2 = \cdots = \sigma_p^2 &= 1 \\
\sigma_{p+1}^2 = \cdots = \sigma_{p+q}^2 &= -1 \\
\sigma_i \sigma_j + \sigma_j \sigma_i &= 0 & \text{if } i \neq j
\end{align*}
\]

Dirac's idea

Set $A = \mathbb{R}[x_1, \ldots, x_{p+q}]$, and

\[X \otimes_A \mathbb{R} \ni \mathcal{E} = \sum_{i=1}^{n} x_i \sigma_i\]

$\mathbb{Z}_2$-graded free $A$-module

\[\mathcal{E}^2 = \sum_{i,j} x_i x_j \sigma_i \sigma_j\]

\[= \sum_i x_i^2 \sigma_i^2\]

\[= x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2\]

acting on $X \otimes_A \mathbb{R}$
Potentials  Let \( R = k[x_1, \ldots, x_n] \) be a commutative \( \mathbb{Q} \)-algebra, then \( f \in R = k[x_1, \ldots, x_n] \)

is called a potential if

1. \( \partial x, f, \ldots, \partial x, f \) is quasi-regular

2. \( R / (\partial x, f, \ldots, \partial x, f) \) is a f.g. free \( k \)-module

3. the Koszul complex of \( \partial x, f, \ldots, \partial x, f \) is exact outside \( \text{deg. 0} \).

Example  \( f \in C[x_1, \ldots, x_n] \) such that \( \dim \mathbb{C} C[x_1, \ldots, x_n] / (\partial x, f, \ldots, \partial x, f) < \infty \).

(isolated critical points)

Defn  The DG\( \mathcal{A} \)-category \( \mathcal{A} = mf(R, f) \) has

- objects f.rank matrix factorisations of \( f \), i.e. \( X \in \{ d \}^3 = f \cdot 1_X \).

- morphisms \( \mathcal{A}(X, Y) = \{ \text{Hom}_R(X, Y), \alpha \mapsto dyx - (-1)^{\alpha} \alpha dx \} \).

This is a \( \mathbb{Z}_2 \)-graded DG\( \mathcal{A} \)-category over \( R \).
Remarks

• $\text{hmf}(R, f) := H^0 \text{mf}(R, f)$ is triangulated (Calabi-Yau).

• Given a quadratic space $V$ with associated quadratic $f \in \text{Sym}(V^*)$

\[
\text{Mod}_{f, \text{d}} \mathbb{C}(V) \cong \text{hmf}(\text{Sym}(V^*), f)^{\infty}
\]

(Buchweitz-Eisenbud-Herzog)
Remarks

• $\text{hmf}(R, f) := H^0\text{mf}(R, f)$ is triangulated (Calabi-Yau).

• Given a quadratic space $V$ with associated quadratic $f \in \text{Sym}(V^*)$

  $\text{Mod}_{f,a}^\mathbb{Z}_2 C(V) \cong \text{hmf}(\text{Sym}(V^*), f)^\omega$

  (Buchweitz-Eisenbud-Herzog)

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From a potential $f$ to an $A_\infty$-algebra $A_f$

Assume $k$ is a field and $\text{Sing}(f) = \{0\}$. Then there is a standard generator

$$\text{thick}(A) = \text{hmf}(R, f)^\omega$$

$$\text{perf End}_R(A) \cong \text{hmf}(R, f)^\omega$$

$$(Keller-Lefevre)$$

$$\text{perf}_\infty H^*\text{End}_R(A) \cong \text{hmf}(R, f)^\omega$$

$A_\infty$-algebra $A_f$, is a Clifford algebra for quadratic $f$.

$A_\infty$-products package higher derivatives of $f$. 

$A_\infty$-transfer (minimal model theorem)
Let $\text{Crit}_R$ be the bicategory of $A_\infty$-algebras $A(f,\pm)$ associated to isolated critical points and their $A_\infty$-bimodules ($\cong$ the bicategory $\mathcal{L}G_{1R}$).

**Theorem** (Carqueville–Montoya ’18) $\text{Crit}_R$ is a symmetric monoidal bicategory in which every object is fully dualisable, and therefore determines an extended 2D framed TFT

$$\text{Bord}_{2,1,0}^{\text{fr}} \to \text{Crit}_R.$$

Moreover $\text{Crit}_R^{\text{ndg}} \subset \text{Crit}_R$.

essentially due to Buchweitz–Eisenbud–Herzog.
\[ f = \frac{1}{2} x_1^2 - \frac{1}{2} x_2^2 \]

Phase portrait

\[ f = \frac{1}{3} x_1^3 - \frac{1}{2} x_2^2 \]

degenerate critical point

\[ \dot{x}_1 = x_1^2 \]
\[ \dot{x}_2 = -x_2 \]

\[ \text{Question: What algebra to associate to } (f, x^*)? \]

- reduce to \( C(T_{x^*} U, H_f|_{x^*}) \) in the nondeg. case
- form a symmetric monoidal bicategory
II. Constructing $\mathcal{A}_\infty$-categories

Throughout $k$ is a commutative $\mathbb{Q}$-algebra and $W \in R = k[x_1, \ldots, x_n]$ a potential

**Question** What is the geometric content of the $\mathcal{A}_\infty$-products on $\text{hmf}(R, W)$? (not just the generator)

**References**


Preliminaries

Defn. A small $\mathbb{Z}_2$-graded $\infty$-category $\mathcal{B}$ over $k$ has a set $\text{ob}(\mathcal{B})$ of objects, and $\mathbb{Z}_2$-graded $k$-modules $\mathcal{B}(a, b)$ for all $a, b \in \text{ob}(\mathcal{B})$ equipped with suspended forward compositions which are odd linear maps

$$r_{a_0, \ldots, a_n} : \mathcal{B}(a_0, a_1)[1] \otimes \cdots \otimes \mathcal{B}(a_{n-1}, a_n)[1] \to \mathcal{B}(a_0, a_n)[1]$$

satisfying the $\infty$-constraints (without explicit signs)

$$\sum_{i \geq 0, j \geq 1, 1 \leq i + j \leq n} r_{a_0, \ldots, a_i, a_{i+j}, \ldots, a_n} \circ (\text{id}_{a_0 a_i} \otimes \cdots \otimes r_{a_i, \ldots, a_{i+j}} \otimes \cdots \otimes \text{id}_{a_{n-1} a_n}) = 0$$

Example. Any $\mathbb{Z}_2$-graded DG-category, $r_n = 0$ for $n \geq 3$. 
Finite $A_\infty$-model

Let $\mathcal{F}: k \to R$ be a morphism of commutative rings, $\mathcal{A}$ a $D\mathcal{A}$-category over $R$.

Restriction of scalars gives a functor

$$
\begin{array}{ccc}
A_\infty\text{-}\text{cat}(R) & \xrightarrow{\mathcal{F}_*} & A \\
\downarrow & & \downarrow \\
A_\infty\text{-}\text{cat}(k) & \xrightarrow{\mathcal{F}_*(\mathcal{A})} & \mathcal{B}
\end{array}
$$

Def. A finite $A_\infty$-model of $\mathcal{A}$ over $k$ is an $A_\infty$-category $\mathcal{B}$ over $k$ with all Hom-spaces f.g. projective over $k$, $A_\infty$-functors $F, G$ and $A_\infty$-homotopies $F \circ G \simeq 1$, $G \circ F \simeq 1$. 

\[ \text{may have } r_i \neq 0 \]
Minimal $A_\infty$-model

Let $\phi: k \to R$ be a morphism of commutative rings, $A$ a DG-category over $R$. Restriction of scalars gives a functor

$$
\begin{array}{ccc}
A_\infty\text{-cat}(R) & \xrightarrow{\phi_*} & A \\
\downarrow & & \downarrow \\
A_\infty\text{-cat}(k) & \xrightarrow{\phi_*} & \phi_*(A) \\
\end{array}
$$

$$
\xrightarrow{\mathcal{F}}
$$

$$(H^*(A), \{r_n\}_{n \geq 2})
$$

**Def** A minimal $A_\infty$-model of $A$ over $k$ is an $A_\infty$-structure $\{r_n\}_{n \geq 1}$ on $H^*(A)$ with $r_1 = 0$, $r_2$ induced by composition, and $A_\infty$-functors $F, G$ and $A_\infty$-homotopies $F \circ G \cong 1$, $G \circ F \cong 1$.

Let $f: k \to R$ be a morphism of commutative rings, $\mathcal{A}$ a $\mathcal{D}\mathcal{A}$-category over $R$

Restriction of scalars gives a functor

$$
\begin{array}{ccc}
A_\infty\text{-}cat(R) & \xrightarrow{f_\ast} & A_\infty\text{-}cat(k) \\
\downarrow & & \downarrow f_\ast \\
A_\infty\text{-}cat(f) & \xrightarrow{f_\ast(A)} & B \xleftarrow{C} E
\end{array}
$$

may have $r_i \neq 0$.

**Def.** An idempotent finite $A_\infty$-model of $\mathcal{A}$ over $k$ is an $A_\infty$-category $B$ with all Hom-spaces f.g. projective over $k$, $A_\infty$-functors $F, G, E$ as above and $A_\infty$-homotopies $F \circ G \simeq E, C \circ F \simeq 1$. ($E = 1$ gives finite models)
Why finite models?

- **Idempotent finite model**
  \[
  (\beta, E_1, E_2, \ldots, r_1, r_2, r_3, \ldots)
  \]
  \[A_{\alpha}\text{-cat}(k)^\infty\]
  \[
  (A, 1, r_1, r_2)
  \]

- **Finite model**
  \[
  (\beta, r_1, r_2, r_3, \ldots)
  \]
  \[
  (A, r_1, r_2)
  \]

- **Minimal model**
  \[
  (H^*(A), r_2, r_3, \ldots)
  \]
  \[
  (A, r_1, r_2)
  \]

- String field theory \((A_\infty)\) vs. topological field theory \((\Delta\text{ed})\).
  \[
  (H^*(A), r_2, r_3, \ldots)
  \]
  \[
  (H^*(A), r_2)
  \]

- The information in higher products is important (e.g. for studying moduli).

The question is: which kind of finite model best packages this information?

Physics refs. Lazaro \(\text{(JHEP 2001)}, \text{Lazaro-\text{-}Roiban} \text{(JHEP 2002)}\), Lazaro \(\text{(2006)}\), Carqueville-\text{-}Dowdy-\text{-}Recknagel \(\text{(JHEP 2012)}\), Carqueville-\text{-}Kay \(\text{(CMP 2012)}\), Baumgartl-\text{-}Brunner-\text{-}Gaberdiel \(\text{(JHEP 2007)}\), Baumgartl-\text{-}Wood \(\text{(JHEP 2009)}\), Knapp-\text{-}Omer \(\text{(JHEP 2006)}\).
idempotent finite model

\( (\beta, E_1, E_2, \ldots, r_1, r_2, r_3, \ldots) \)

\( (A, 1, r_1, r_2) \)

minimal model

\( (H^*(A), r_2, r_3, \ldots) \)

\( (A, r_1, r_2) \)
idempotent finite model

\((\beta, E_1, E_2, \ldots, r_1, r_2, \ldots)\)

12

\((A, 1, r_1, r_2)\)

\(k\) a field

minimal model

\((H^\ast(A), r_2, r_3, \ldots)\)

12

\((A, r_1, r_2)\)

Choose \(k\)-linear homotopy equivalences

\[
\begin{array}{ccc}
A(a, b) & \xrightarrow{f} & H^\ast A(a, b) \\
\downarrow g & & \downarrow f \\
1 - [dA, H] & & 1
\end{array}
\]

and transfer \(A\infty\)-structure to \(H^\ast(A)\)

• useful for special objects (e.g. \(k^{\text{stab}}\))
  (Seidel, Dyckerhoff, Efimov, Sheridan)

• depends on \(k\) being a field.
**Idempotent finite model**

\((β, E_1, E_2, ..., r_1, r_2, r_3, ...)\)

\[(A, 1, r_1, r_2)\]

- Exists for all of \(A = mf(W)\)
- Constructive when Gröbner methods are available (e.g. \(k\) a field or poly. ring).
- Downside: not minimal. However, we know TFT formulae (HRR, Kapustin-Li) can be derived directly from \(β, E_1\).
- For special objects can split \(E\).
- **Key point**: first enlarge \(A!\)

**Minimal model**

\((H^*(A), r_2, r_3, ...)\)

\[(A, r_1, r_2)\]

Choose \(k\)-linear homotopy equivalences

\[A(a, b) \xrightarrow{f} H^*A(a, b)\]

and transfer \(A_\infty\)-structure to \(H^*(A)\)

- useful for special objects (e.g. \(k^{stab}\)) (Seidel, Dyckerhoff, Efimov, Sheridan, Tu)
- depends on \(k\) being a field.
An idempotent finite $A_\infty$-model of $mf$

\[ A = mf(R, W) \quad R = k[x_1, \ldots, x_n] \quad t_i = \partial x_i, W, \ldots, t_n = \partial x_n W \]

\[ A_\Theta = \wedge F_\Theta \otimes_R mf(R, W) \otimes_R \hat{R} \quad F_\Theta = k\mathcal{O}_1 \oplus \ldots \oplus k\mathcal{O}_n \]

\[ B = R/I \otimes_R mf(R, W) \]

\[ A \to A \otimes_R \hat{R} \quad \xrightarrow{\text{homotopy equiv.}} \quad A_\Theta \quad \xleftarrow{\text{homotopy equiv.}} \quad B \]

\[ \begin{aligned}
A_\Theta & \quad \xrightarrow{\alpha} B \\
\text{e} & \quad \xmapsto{e} \quad E \quad \xleftarrow{\text{\text{Fe}C}} \\
\text{e}(\Theta) & = 0
\end{aligned} \]

\[ \text{Theorem} \quad (B, E) \text{ is an idempotent finite } A_\infty \text{-model of } A \otimes_R \hat{R}. \]
Connections and Residues

Let $k$ be a commutative $Q$-algebra, $R$ a $k$-algebra, and $t_1, \ldots, t_n$ a quasi-regular sequence in $R$ such that $R/I$ is f.g. projective over $k$, $I = (t_1, \ldots, t_n)$.

**Lemma** (Formal tubular neighbourhood) Any $k$-linear section $\delta$ of $R \to R/I$ induces an isomorphism of $k[[t_1, \ldots, t_n]]$-modules

$$\delta^* : R/I \otimes_k k[[t_1, \ldots, t_n]] \to \hat{R}$$

defined by

$$(\delta^*)^{-1}(r) = \sum_{M \in \mathbb{N}^n} r_M \otimes t^M$$

where the $r_M \in R/I$ are unique such that in $\hat{R}$ we have

$$r = \sum_{M \in \mathbb{N}^n} \delta(r_M) t^M.$$
Connections and Residues

Let $k$ be a commutative $\mathbb{Q}$-algebra, $R$ a $k$-algebra, and $t_1, \ldots, t_n$ a quasi-regular sequence in $R$ such that $R/I$ is f.g. projective over $k$, $I = (t_1, \ldots, t_n)$.

Upshot: If $k$ is a field, $R = k[x_1, \ldots, x_n]$, $\delta^*$ may be computed by Gröbner methods.

In general,

$$\delta^* : R/I \otimes_k k[t_1, \ldots, t_n] \xrightarrow{\cong} \hat{R} \quad (k[[t]]\text{-linear})$$

$$\sum_{M \in \mathbb{N}^n} \theta(r_M) t^M = r$$

There is a $k$-linear connection $\nabla : \hat{R} \longrightarrow \hat{R} \otimes_{k[[t]]} \Omega^1_{k[[t]]/k}$, and

Theorem (Lipman, M) $\text{Res}_{R/k}\left[ \frac{r \, dr_1 \cdots dr_n}{t_1, \ldots, t_n} \right] = \text{tr}_{R/I}\left( r[\nabla, r_1] \cdots [\nabla, r_n] \right)$
Connections and Residues

\[ \delta^* : R/I \otimes_k \mathbb{k}[t_1, \ldots, t_n] \xrightarrow{\cong} \hat{R} \]

\[ \nabla : \hat{R} \longrightarrow \hat{R} \otimes_{\mathbb{k}[[t]]} \bigoplus_{\mathbb{k}[[t]]} k \]

\[ \text{Res}_{R/k} \left[ \frac{rd_{t_1} \cdots dr_{t_n}}{t_1, \ldots, t_n} \right] = tr_{R/I} \left( r[\nabla, r_1] \cdots [\nabla, r_n] \right) \]
\[
\begin{align*}
A &= m_f(R, W) \cong k[x_1, \ldots, x_n] \quad d_{A}, m_2 \\
A_\otimes &= \Lambda F_\otimes \otimes_k m_f(R, W) \otimes_k \hat{R} \quad d_{A \otimes}, m_2 \\
B &= R/I \otimes_k m_f(R, W) \quad d_{B}, m_1, m_2, \ldots \\
\Lambda A \rightarrow A \otimes_k \hat{R} \xleftarrow{F} \rightarrow A_\otimes \xleftarrow{G} B
\end{align*}
\]

\[
At_A := \begin{bmatrix} \nabla, d_A \end{bmatrix}
\]

(Atiyah class of \( A \))

\[
6_\infty = \sum_{m \geq 0} (-1)^m (\zeta A \Lambda A)^m \zeta : B \rightarrow A_\otimes
\]

\[
\phi_\infty = \sum_{m \geq 0} (-1)^m (\zeta A \Lambda A)^m \zeta \nabla : A_\otimes \rightarrow A_\otimes
\]

\[
\delta = \sum_{m \geq 0} \lambda_i \Theta^* : A_\otimes \rightarrow A_\otimes
\]

\[
At_A, \delta \text{ rewritten using } \mathcal{Z}^* \]

\[
(A_\otimes(x, y), d_{\Lambda A}) \xleftarrow{h.a.} (B(x, y), d_{\Lambda B})
\]

\[
\text{transfer } A_\infty \text{-structure}
\]

\[
A_\otimes(x, y) \cong \Lambda F_\otimes \otimes_k B(x, y) \otimes_k k[x_1, \ldots, x_n] \supset B(x, y)
\]

\[
\nabla = \sum_i \Theta \frac{\partial}{\partial \Theta_i}, \quad \zeta \omega \otimes \varphi \varphi = \frac{1}{|\omega| + 1} \omega \otimes \varphi \varphi
\]

\[
\delta_3 : \beta[1] \otimes 3 \rightarrow \beta[1]
\]
\[ \delta^*: \hat{\mathbb{R}}/\mathbb{I} \otimes_k k[[t_1, \ldots, t_n]] \to \hat{\mathbb{R}} \]
\[ \nabla: \hat{\mathbb{R}} \to \hat{\mathbb{R}} \otimes_{k[[t]]} \Omega^1_{k[[t]]}/k \]

\[ \text{Res}_{\hat{\mathbb{R}}/k} \left[ \frac{r \, dr_1 \cdots dr_n}{t_1, \ldots, t_n} \right] = \text{tr}_{\hat{\mathbb{R}}/\mathbb{I}} \left( r [\nabla, r] \cdots [\nabla, r_n] \right) \]

\[ \mathcal{A}_\theta = \wedge F_0 \otimes \Omega_{\hat{\mathbb{R}}} \]
\[ \equiv \wedge F_0 \otimes \mathcal{B} \otimes k[[t]] \]

\[ r_m^B = r_m^B \left( [\nabla, dA], \lambda_1, \ldots, \lambda_n, \zeta \right) \]
Prove $\mathcal{J} \cong \mathcal{J}'$ by finding generators $G, G'$ and $\mathcal{A}_\infty$-iso $\text{End}(C) \cong \text{End}(C')$.

References


• N. Sheridan, “Homological mirror symmetry for Calabi-Yau hypersurfaces in projective space” Inventiones 2015.


Some optional slides
Proof sketch

Choose homotopies $\lambda_i$ such that $[d\mathcal{A}, \lambda_i] = t_i$.

There is a strict homotopy retraction of complexes over $k$

$$(\mathcal{A}_0(X,Y), d\mathcal{A}) = \left( \Lambda F_0 \otimes_k \text{Hom}_R(X,Y) \otimes_R \hat{R}, d\mathcal{A} \right)$$

$$e^\delta \uparrow \downarrow e^{-\delta}$$

$$\delta = \sum_i \lambda_i \Theta_i$$

$$(\Lambda F_0 \otimes_k \text{Hom}_R(X,Y) \otimes_R \hat{R}, d\mathcal{A} + \sum_i t_i \Theta_i^*)$$

by homological perturbation using connection

$$\nabla$$

using canonical projection

$$(\mathcal{B}(X,Y), \overline{d\mathcal{A}}) = \left( R/I \otimes_R \text{Hom}_R(X,Y), \overline{d\mathcal{A}} \right)$$
Proof sketch

The $A^\infty$-transfer (minimal model) theorem (Kadeishvili, Merkulov, Kontsevich-Soibelman and for our purposes Markl) constructs $A^\infty$-products on $\beta$ and $A^\infty$-homotopy equivalences $F, G$

\[
\begin{align*}
A_\theta & \xrightarrow{F} \beta \\
\Phi^{-1} & \xrightarrow{\text{h.e.}} \Phi & \pi e^{-\delta} \\
(e^\delta \beta_\infty) & \downarrow \Phi & (\beta (x, y), \overline{c_\mathcal{A}}) \\
(\overline{A_\theta (x, y), c_\mathcal{A}}) & \Phi^{-1} \Phi & 1 - \left[ d_\mathcal{A}, H \right]
\end{align*}
\]

\[
F_1 = \Phi, \quad G_1 = \Phi^{-1}, \quad G \circ F \cong 1
\]

$r_1^\beta, r_2^\beta$ induced from $r_1^\mathcal{A}, r_2^\mathcal{A}$.
\[ A = mf(R; W) \Rightarrow k[x_1, \ldots, x_n] \]
\[ A_\theta = F_\theta \otimes_k mf(R, W) \otimes_R \hat{\mathbb{R}} \]
\[ \beta = R/I \otimes_k mf(R, W) \]
\[ A \rightarrow A_\otimes \hat{\mathbb{R}} \rightarrow A_\theta \xrightarrow{f} \beta \]

\[ A_\theta(x, y) = \bigwedge F_\theta \otimes_k \text{Hom}_R(X, Y) \otimes_R \hat{\mathbb{R}} \]

\[ \approx \bigwedge F_\theta \otimes_k \text{Hom}_k(\tilde{X}, \tilde{Y}) \otimes_k \hat{\mathbb{R}} \]

(choose bases for \(X, Y\)

i.e. \(X \cong \tilde{X} \otimes_k R\))

\[ \approx \bigwedge F_\theta \otimes_k \beta(X, Y) \otimes_k \mathbb{k}[t_1, \ldots, t_n] \]

\[ \cup \]

\[ \nabla = \sum_i \Theta_i \frac{\partial}{\partial t_i} \]

\[ \zeta(w \otimes \alpha \otimes f) = \frac{1}{|w| + |f|} w \otimes \alpha \otimes f. \]
Feynman diagrams

Suppose $X = \Lambda F_3 \otimes k R$, $Y = \Lambda F_2 \otimes k R$ are Koszul-type MFs.

$\Lambda( F_0 \oplus F_3^* \oplus F_2 ) \otimes k R/I \otimes k [t \pm 1] \supset \Lambda( F_3^* \oplus F_2 ) \otimes k R/I$

$A_0 (X, Y)$, interior of trees

$B( X, Y )$, exterior

- Apart from $\Omega$, all operators involved in computing $A_\infty$-products can be written as polynomials in creation and annihilation operators.

- Feynman diagrams organise reduction of such trees to normal form.

Example: One contribution for $W = \frac{1}{T} x^j Y$ to

$\lambda_3 : \beta (x, x) [1] \otimes \beta (x, y) [1] \otimes \beta (y, y) [1] \rightarrow \beta (x, y) [1]$

$\lambda_3 ( x^2 y \otimes x y z \otimes x^2 y z^* )$