Introduction to the LG/CFT correspondence

The aim of this lecture is to sketch the Landau-Ginzburg (LG) / Conformal Field Theory (CFT) correspondence, which has its origin in mathematical physics but which has inspired interesting pure mathematics. The correspondence is far from fully understood, mathematically or physically. On the LG side the main mathematical ingredient is the theory of matrix factorisations, and we begin with a (slightly nonstandard) introduction to this theory, which we hope is helpful in trying to understand the relation to vertex algebras via Drinfeld-Sokolov reduction.

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Outline

1. Matrix factorisations & LG models
2. Simple singularities and minimal models in CFT
3. The LG/CFT correspondence
One way to understand the theory of matrix factorisations is as a generalisation (in a sense that can be made precise) of the theory of Clifford algebras, so let us begin with a reminder on the latter. Let $k$ be a field, $\text{char } k = 0$.

**Definition.** Given a f.d. vector space $V$ and symmetric bilinear form $B: V \times V \to k$, the Clifford algebra of $(V, B)$ is the $\mathbb{Z}_2$-graded algebra

$$
\text{Cl}(V, B) := \frac{T(V)}{(v \otimes w + w \otimes v - 2B(v, w))_{v, w}}
$$

(i.e. generated by a basis $e_1, \ldots, e_n$ of $V$ subject to $e_i e_j + e_j e_i = 2B(e_i, e_j)$)

**Example** $\text{Cl}(V, 0) \cong \bigwedge V$ as $\mathbb{Z}_2$-graded algebras.

**Lemma** There is always an iso of $\mathbb{Z}_2$-graded vector spaces

$$
\bigwedge V \xrightarrow{\cong} \text{Cl}(V, B)
$$

$\Rightarrow$ $\text{Cl}(V, B)$ is a deformation (as an algebra) of $\bigwedge V$.

**Question** Are there other deformations of $\bigwedge V$ as a $\mathbb{Z}_2$-graded algebra?

The question is best phrased in terms of the Hochschild cohomology $\text{HH}^*_{\mathbb{Z}_2}(\bigwedge V)$ (in it $\mathbb{Z}_2$-graded form) of $\bigwedge V$, or what is the same, the deformations of $\bigwedge V$ within the space of $\mathbb{Z}_2$-graded $A_{\infty}$-algebras ( = vector spaces $A$ equipped with operations $m_n: A^{\otimes n} \to A$ for $n \geq 1$).

$$
\{ \text{Z}_2\text{-graded algebras} \} \subset \{ \text{Z}_2\text{-graded DG-algebras} \} \subset \{ \text{Z}_2\text{-graded } A_{\infty}\text{-algs} \}
$$
Yes. Say $V = \mathbb{C}^n$. Every polynomial $W \in \mathbb{C}[z_1, \ldots, z_n]$ with an isolated singularity at $0$ (i.e. $W(0) = 0$ and $\dim \mathbb{C}[z]/(\partial_1 W, \ldots, \partial_n W) < \infty$) gives rise to an $A_\infty$-algebra

$$A_W = \left( \Lambda^{C^n}, m_2, m_3, \ldots, m_n, \ldots \right)$$

whose underlying vector space is $\Lambda^{C^n}$ (existence due to Dyckhoff).

**Example**: $W = \sum_{i=1}^n z_i^2$ is the quadratic form of $B : \mathbb{V} \times \mathbb{V} \to \mathbb{C}$, $B(e_i, e_j) = \delta_{ij}$, and $A_W \cong C(r, B)$ as $A_\infty$-algebras

**Example**: $n=1$, so $\Lambda^{C} = \mathbb{C}[z]/z^2$, for $W = z^N \in \mathbb{C}[z]$ there is a Hochschild cocycle ($N > 3$)

$$c \in HH^N(\Lambda C), c : (\Lambda C)^{\otimes N} \to \Lambda C$$

$$c(z \otimes \cdots \otimes z) = 1 \quad \text{(zero on other basis elt.)}$$

The corresponding deformation of $\Lambda C$ is the $A_\infty$-algebra

$$A_{2^N} = \left( \Lambda C, m_2, 0, \ldots, 0, c, 0, \ldots \right)$$

with product on $\Lambda C$. 
The above raises the question: what are the categories of $A_\infty$-modules over these deformations? And how are the $m_n$ derived from $W$?

\[ \text{per}_\infty(A^w) = ? \]
\[ m_n = ? \]

The first question is best-addressed using matrix factorisations (finally), but the second is beyond the scope of this talk (and only recently worked out).

**Def** Let $W \in \mathbb{C}[z_1, \ldots, z_n]$. A matrix factorisation $X$ of $W$ is described equivalently as either

(i) a pair $A, B \in \text{Mk}(\mathbb{C}[z])$ such that $AB = BA = W \cdot 1_k$. 
(ii) a free $\mathbb{Z}_2$-graded $\mathbb{C}[z]$-module $X = X^0 \oplus X^1$ with odd $d_X : X \to X$ (the differential) s.t. $d_X^2 = W \cdot 1_X$.

**Def** A morphism $\phi : (X, d_X) \to (Y, d_Y)$ is degree zero, $\mathbb{C}[z]$-linear map $\phi : X \to Y$ with $d_Y \phi = \phi d_X$. We say $\phi \sim \gamma$ are homotopic if there is $h : X \to Y$ odd with $\phi - \gamma = d_Y h + h d_X$.

**Def** $\text{hmf}(W) := \{ \text{matrix factorisations with homotopy equivalence classes of morphisms} \}$
Theorem (Eisenbud, Buchweitz-Orlov, Auslander) $\text{hmf}(W)$ is a Calabi-Yau triangulated category and

$$\text{hmf}(W) \cong \mathbb{D}^b(\text{coh } Z(W))/\text{perf } Z(W)$$

where $Z(W) = \{ a \in \mathbb{C}^n | W(a) = 0 \}$.

Theorem (Dyckerhoff) There exist linear maps $m_k : (\Lambda \mathbb{C}^n)^\otimes_k \to \Lambda \mathbb{C}^n$ such that $(\Lambda \mathbb{C}^n, \{ m_k \}_{k \geq 2})$ is an $A\infty$-algebra and

$$\text{perf}(\mathfrak{A}_W) \cong \text{hmf}(W)$$

Example For $W = \sum_i z_i^2$, $\text{per}_{\infty}(\mathfrak{A}_W) \cong \text{f.d. representations of } C1(V, B)$, as $\mathfrak{A}_W = C1(V, B)$ (recover old result of Buchweitz-Eisenbud-Heizog).

Example $W = \mathbb{Z}^N$, $d_x = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix} = \begin{pmatrix} 0 & z_i \\ z_{N-i} & 0 \end{pmatrix}$, $1 \leq i \leq N$.

i.e. $\text{per}_{\infty}(\mathfrak{A}_{\mathbb{Z}^N}) \cong \text{hmf}(\mathbb{Z}^N)$, (with $m_k$ as defined above).

related to $A_{N-1}$ quiver, since

$$\mathbb{D}^b(\text{rep } A_{N-1})$$

$\text{hmf}^{\text{gr}}(\mathbb{Z}^N+x^2+y^2)$
Landau–Ginzburg models

$LG_w$ is an $N=2$ supersymmetric QFT whose Lagrangian involves the polynomial $W \in \mathbb{C}[z_1, \ldots, z_n]$. The topological $B$-twist of $LG_w$ is described in various aspects by rigorous mathematics:

**Physics**

- Bulk observables
  \[ J_w = \frac{C[z_1, \ldots, z_n]}{(\partial z_1 W, \ldots, \partial z_n W)} \]

- Boundary conditions

  (similar to the role of vector bundles in nonlinear sigma models)

- Boundary observables

  \[ \text{bosonic } \text{Hom}(X, Y) \text{ in } \text{Hom}(X, Y[1]) \]

**Math**

- Defect conditions

  (may be composed \( \Rightarrow \otimes \)-structure)

\( \vdots \)
Move precisely:

\[
\begin{array}{c}
\text{Open-closed 2D TFT} & \leftrightarrow \text{Calabi-Yau triangulated category} \\
\text{L}_W & \text{hmf}(W)
\end{array}
\]

\[
\begin{array}{c}
\text{2D defect TFT} & \leftrightarrow \text{(conjecturally) Pivotal bicategory} \\
\{\text{L}_W\}_W & \text{L}_{\mathcal{G}}
\end{array}
\]

Lazanoiu-McNamee, Carqueville-Runkel, Carqueville-N

2) Simple singulartion and minimal CFTs

Given \( W \in \mathbb{C}[\bar{z}_1, \ldots, \bar{z}_n] \) topology \( \mathbb{C}[\bar{z}] \) such that \( \mathbb{C}[\bar{z}] \rightarrow \mathbb{C}[\bar{z}]/\mathbb{H}^k \) is continuous for all \( k > 1 \). Then provided

\[
f = \text{germ}_W W \in \mathbb{m} \subseteq \mathbb{C}[\bar{z}]
\]

defines an isolated singularity there is an open neighborhood \( f \in U \subseteq \mathbb{m} \) in which every pt. is either smooth or also an isolated singularity. We say \( f \) is a simple singularity if \( \exists U \) containing only finitely many "distinct" singularities \( (= \text{orbit of } \text{Aut} \mathbb{C}[\bar{z}] \). )

Theorem (Arnold '72) The simple singularities have an ADE classification:

\[
\begin{align*}
A_k & : x_1^{k+1} + x_2^2 + \cdots + x_n^2 \\
D_k & : x_1 (x_2^2 + x_1^{k-2}) + x_3^2 + \cdots + x_n^2 \\
E_6 & : x_1^2 + x_2^4 + \text{squares} \\
E_7 & : x_1 (x_2^2 + x_3^3) + \text{squares} \\
E_8 & : x_1^3 + x_2^5 + \text{squares}
\end{align*}
\]
• 2D conformal field theories are CFTs which are covariant w. r. t. local conformal transformations. Infinitesimally these transformations are generated by two copies of the Virasoro algebra (called the left and right chiral algebras). The central charge of the CFT is the $c$ appearing in these Virasoro algebras. A given CFT may have an extended chiral algebra at containing Virasoro, such as affine Kac-Moody algebras or the superconformal algebras.

• An $N=2$ superconformal field theory is a CFT whose chiral algebra contains a particular Lie superalgebra called the “$N=2$ superconformal algebra”. The minimal $N=2$ super CFTs have an ADE classification, see [ADE], with e.g.

$$\left[ A_k - \text{type } N=2 \text{ minimal CFT, } \ c = 3 - \frac{6}{k+1} \right]$$

(by coset construction $\frac{\widehat{sl}(2)_{d-2} \times \widehat{su}(1)_{4}}{\widehat{su}(1)_{2d}}$)

Aside ————

Given $W \in \mathbb{C}[x_1, \ldots, x_n]$ and a decomposition $W = \sum_{i=1}^{k} a_i b_i$, we have the Koszul factorisation of $W$:

$$X = \left( \Lambda(C_{y_1} \otimes \cdots \otimes C_{y_k}) \otimes C[z], \sum_{i} a_i \gamma_i^* + \sum_{i} b_i \gamma_i \right)$$

These are the most commonly encountered MFs in practice (e.g. in knot homology, LG/CFT correspondence, ...) and the underlying vector space of the corresp. $A \otimes \mathbb{C}$-module is again an exterior algebra.
3) **The LG|CFT correspondence** (based on [R])

Based on:

- **UV** → **IR** via **RG flow**
  - natural length scale → ∞

**N=2 LG model**
- with potential \( W \)
  - (e.g. \( x_1^{R_1}x_2^{R_2}x_3^{R_3} \))

**N=2 SCFT**
- \( c = 3 \sum_{i} (1 - q(x_i)) \)
  - (e.g. \( A_b \) minimal model)

**Preservation of topological quantities under RG flow “implies”**

**[LG side]** \[\equiv\] **[CFT side]**

**Bulk observables** \( J_w \) \[\equiv\] **Chiral primary fields**

**Martinec '89, Vafa, Warner '89, Lenhe, Vafa, Warner '89**

**(B-type) Boundary conditions**

\[ \mathcal{A} \stackrel{\text{Kapustin-Li '02,'03}}{\equiv} \mathcal{A}' \]

- (MFs of \( W \))

\[ \mathcal{A}_b \]

- (reps. of chiral algebra, SVOA)

**Boundary conditions**

\[ \mathcal{A} \equiv \mathcal{A}' \]

**Defect conditions**

\[ \mathcal{B} \stackrel{\text{Brunner-Roggenkamp'07}}{\equiv} \mathcal{B}' \]

- (MFs of \( W_1-W_2 \))

\[ \mathcal{B}_b \]

- (bimodules over chiral algebra)

**Defect conditions**

\[ \mathcal{B} \equiv \mathcal{B}' \]

**Defect conditions**

\[ \mathcal{B} \equiv \mathcal{B}' \]

- (natural geometric defects between \( A \)-type singularities correspond to defects/flows between minimal CFTs).
**Theorem (Davydov-Ros Camacho-Runkel '14)** Let $k$ be even. There is an equivalence of tensor categories

$$\mathcal{P}^{\text{gr}}_{k+1} \xrightarrow{\sim} \mathcal{C}(N=2, k+1)_{\text{NS}} \quad (9.1)$$

\begin{align*}
\{ \text{graded MFs of } y^{k+1} - x^{k+1} \} & \quad \{ \text{irred. highest weight reps. of } \\
\text{(i.e. defects between two copies of } \text{Ak-type minimal SVOA) } \}
\end{align*}

Example $\mathcal{P}^{\text{gr}}_{k+1}$ is generated by

$$P_{a, \lambda} = \begin{pmatrix}
\frac{a+\lambda}{2} & \prod_{i=\alpha}(y-\gamma^i x) \\
\prod_{i=\alpha}(y-\gamma^i x) & 0
\end{pmatrix}$$

using $y^{k+1} - x^{k+1} = \prod_{i=\alpha}(y-\gamma^i x)$

So far all rigorous checks of the LG/CFT correspondence are “bespoke” (e.g. (9.1) is proved by a direct comparison of both sides).

**OPEN QUESTION**: What is a conceptual mathematical explanation/proof of the LG/CFT correspondence? of the kind

$$\{ \text{geometry of singular hypersurface } W = 0 \} \quad \longrightarrow \quad \{ \text{super VOA} \}$$

**Summary**: A simple singularity $W(x_1, x_2, x_3)$ gives rise to

$$\text{rep}_\infty(\land C^3, m_2, m_3, \ldots) \cong \text{hmf}(W) \quad \longleftrightarrow \quad \text{Rep}(V)$$

\[ \text{LG/CFT} \]

$A_\infty$-algebra, defn of exterior algebra, $A_\infty$-modules

$N=2$ super VOA with modular invariants of the ADE type of $W$
Some observations

- Following [C90], we can try to construct representations of the $N=2$ superconformal algebra using Hochschild cocycles of exterior algebras. Recall the loop algebra of a Lie algebra $g$ is $g \otimes C^\infty(S^2)$ where $C^\infty(S^2)$ are smooth complex-valued functions. If we pass to Fourier coefficients and ignore convergence (Q: does this change the cyclic cohomology?) this is

$$g \otimes C[[t, t^{-1}]].$$

The super-loop algebra is $(G_{N-1} = \bigwedge C^N$ the exterior algebra)

$$g \otimes C[[t, t^{-1}]] \otimes C G_{N-1}.$$ (*)

Kac-Moody (resp. super Kac-Moody) Lie algebras have underlying vector space $(g \otimes C[[t, t^{-1}]]) \otimes C R$ with $R$ central and the new bracket determined by a Lie algebra 2-cocycle on $g \otimes C[[t, t^{-1}]]$ call it $\omega$, by

$$[(\bar{x}, \alpha), (\gamma, \beta)]_{\text{new}} = \left[\bar{x}, \gamma\right]_g + \omega(\alpha, \beta)$$

(for the super cone, we need a $\mathbb{Z}_2$-graded cocycle on (*)). The super cone is richer, since there are more relevant cocycles to twist by:

$$\left\{\text{cyclic Hochschild 1-cocycles of } G_{N-1}\right\} \twoheadrightarrow \left\{\text{cyclic Lie 2-cocycles of } C[[t, t^{-1}]][\otimes G_{N-1}]\right\}$$

large!
Let $C$ be a Lie superalgebra, and

$$\text{Dev}(C) = \{ \mathcal{L} : C \rightarrow C \mid \mathcal{L} \text{ is } E\text{-linear, homog. and for all } x, y \in C \}
\mathcal{L}(\{x, y\}) = [\mathcal{L}x, y] + (-1)^{\|x\|\|y\|} [x, \mathcal{L}y]$$

This is a Lie superalgebra with the bracket from $\text{End}_{E}(C)$. Given a $\mathbb{Z}_2$-graded 2-cocycle on $C$, we can form the central extension $C \oplus \mathbb{R}$ with the bracket $[-,-]_\omega$ from the cocycle $\omega$, and given $\mathcal{L} \in \text{Dev}(C)$ ask if $\tilde{\mathcal{L}} : C \oplus \mathbb{R} \rightarrow C \oplus \mathbb{R}$ defined by $\tilde{\mathcal{L}}|_C = \mathcal{L}$, $\tilde{\mathcal{L}}|_{\mathbb{R}} = 0$ is a graded derivation for the new bracket. This amounts to

$$\omega(\tilde{\mathcal{L}}x, y) + (-1)^{\|x\|\|y\|} \omega(x, \tilde{\mathcal{L}}y) = 0 \quad \forall x, y.$$ (X)

$\text{Def}$ $\text{Dev}_{\omega}(C)$ is the Lie subalgebra of $\text{Dev}(C)$ given by all $\mathcal{L}$ satisfying (X).

The way [C93] define Hochschild cohomology is to identify $n$-cocycles of $A$ as $E$-linear forms $A \otimes A^n \rightarrow A$, probably as follows: we would say $n$-cocycles are $E$-linear $\tilde{A} \otimes \tilde{A} \rightarrow A$, i.e. $A^* \otimes A^* \rightarrow A$, and if $A$ is self-dual, this is the same as $A^* \rightarrow A$.

$\text{Theorem}$ For $N=1$ there exists a cyclic Hochschild cocycle $\alpha$ with associated Lie algebra 2-cocycle $\omega$ of $C = C[[t, t^{-1}]] \otimes \mathbb{C} G_\text{n}$, such that $\text{Dev}_{\omega}(C)$ is the $N=1$ superconformal algebra.

$\text{Theorem}$ For $N=2$ there is $\alpha, \omega$ with $\text{Dev}_{\omega}(C)$ containing a copy of the $N=2$ superconformal algebra.

$\square$ role for higher Hochschild cocycles of the Grassmann alg. $C_n$, via $\text{Loo}$-algebras? related by [C95] to a degenerate LG model. ($j=-1 \Rightarrow$ central change $-1$)
References


