Generalised orbifolding of simple singularities III

In this lecture we finally introduce the two examples of bicategories of interest, the bicategories of Landau-Ginzburg models and its graded version. One of the aims here is to be a concrete as possible, so we describe all structure maps.

Def^n The bicategory $\mathcal{LG}$ has

- objects are potentials, i.e. pairs $(\mathbb{C}[x_1,\dots,x_n], W)$ with $W \in \mathbb{C}[\underline{x}]$ having isolated critical pt., i.e. $J_W = \mathbb{C}[\underline{x}]/(\partial_{x_1}W,\dots,\partial_{x_n}W)$ is finite dimensional.

- morphisms $\mathcal{LG}\big((\mathbb{C}[\underline{x}],W),(\mathbb{C}[\underline{y}],V)\big) := \operatorname{hmf}(\mathbb{C}[\underline{x},\underline{y}], V-W)^{\omega}$

- composition there is a functor, say for $U \in \mathbb{C}[\underline{z}]$,

$$\begin{tikzcd}
\mathcal{LG}(V,U)\times\mathcal{LG}(W,V) \arrow[rr, "*"] \arrow[d, phantom, "\shortparallel"] & & \mathcal{LG}(W,U) \arrow[d, phantom, "\shortparallel"] \\
\operatorname{hmf}(\mathbb{C}[\underline{y},\underline{z}],U-V)^{\omega}\times\operatorname{hmf}(\mathbb{C}[\underline{x},\underline{y}],V-W)^{\omega} \arrow[rr] \arrow[d, phantom, "\ni"] & & \operatorname{hmf}(\mathbb{C}[\underline{x},\underline{z}],U-W)^{\omega} \arrow[d, hook, "I"', "\text{fully faithful}"] \\
(Y,X) \arrow[r, mapsto] & \big(Y\otimes_{\mathbb{C}[\underline{y}]}X,\ d_Y\otimes 1+1\otimes d_X\big)\in & \operatorname{HMF}(\mathbb{C}[\underline{x},\underline{z}],U-W)
\end{tikzcd}$$

infinite rank MF

Proposition $Y\otimes_{\mathbb{C}[\underline{y}]}X$ lies in the image of I.

Def^n $Y * X :=$ representative in $\operatorname{hmf}(U-W)^{\omega}$ for $Y\otimes_{\mathbb{C}[\underline{y}]}X$.

To describe morphisms into and out of $Y*X$ it is still convenient to use $Y\otimes X$.

- unit at $(\mathbb{C}[x_1,\dots,x_n], W)$ is defined by $(1 \otimes i \,|= 1)$

$$f \in \mathbb{C}[\underline{x}] \otimes \mathbb{C}[\underline{x}] \qquad {}^{t_i}f := f\big|_{x_i \otimes 1 \mapsto 1 \otimes x_i}$$

$$\partial_{[i]} f := \frac{{}^{t_1 \cdots t_{i-1}}f - {}^{t_1 \cdots t_i}f}{x_i \otimes 1 - 1 \otimes x_i} \qquad \Big(\partial_{[i]} f \Big|_{\substack{x_i \otimes 1 \mapsto 1 \otimes x_i \\ \text{all } i}} = \partial_{x_i} f \Big)$$

$$\Delta_W := \Big(\bigwedge \bigoplus_{i=1}^{n} (\mathbb{C}[\underline{x}] \otimes \mathbb{C}[\underline{x}])\theta_i \;,\; \sum_i \partial_{[i]} W \theta_i + \sum_i (x_i \otimes 1 - 1 \otimes x_i)\theta_i^* \Big)$$

$$d_{\Delta_W}^2 = W(\underline{x} \otimes 1) - {}^{t_1}W + {}^{t_1}W - \cdots - W(1 \otimes \underline{x})$$
$$= W \otimes 1 - 1 \otimes W$$

<u>Def^n</u> Let $\pi : \Delta_W \longrightarrow \mathbb{C}[\underline{x}]$ be the $\mathbb{C}[x]^{\otimes 2}$-linear $\big(\theta_I = \theta_{i_1} \cdots \theta_{i_k} \quad I = \{i_1, \dots, i_k\} \big)$

$$\pi(\theta_I) = 0, \quad I \neq \phi \quad \text{and} \quad \pi(1) = 1.$$

<u>Def^n</u> Given $X \in \mathcal{LG}\big((\mathbb{C}[\underline{x}], W), (\mathbb{C}[\underline{y}], V) \big)$ define

$$\rho_X : X \otimes_{\mathbb{C}[x]} \Delta_W \xrightarrow{1 \otimes \pi} X \otimes_{\mathbb{C}[x]} \mathbb{C}[x] \cong X$$

$$\lambda_X : \Delta_V \otimes_{\mathbb{C}[y]} X \xrightarrow{\pi \otimes 1} \mathbb{C}[y] \otimes_{\mathbb{C}[y]} X \cong X.$$

<u>Proposition</u> Thus defined $\mathcal{LG}$ is a bicategory (triangulated)

<u>Remark</u> $\mathbb{1} := (\mathbb{C}, 0) \in \mathcal{LG}$ and $\mathcal{LG}(\mathbb{1}, W) = hmf(W)^{\omega}$.
This defines a pseudofunctor $\mathcal{LG}(\mathbb{1}, -) : \mathcal{LG} \longrightarrow \underline{\mathrm{Cat}}$
to the bicategory of small categories, $W \longmapsto hmf(W)^{\omega}$ and

sending $X : W \longrightarrow V$ to the functor

$$\begin{array}{ccc} \mathcal{LG}(\mathbb{1}, W) & \xrightarrow{X * -} & \mathcal{LG}(\mathbb{1}, V) \\ \cong & & \cong \\ \mathrm{hmf}(W)^{\omega} & & \mathrm{hmf}(V)^{\omega} \end{array}$$

and a morphism $\varphi : X \longrightarrow X'$ of MFs to a natural transformation

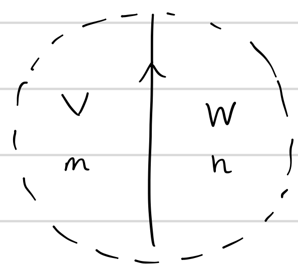
$$\begin{tikzcd} \mathrm{hmf}(W)^{\omega} \arrow[r, bend left, "X * -"] \arrow[r, bend right, "X' * -"'] \arrow[r, phantom, "\Downarrow \varphi"] & \mathrm{hmf}(V)^{\omega} \end{tikzcd}$$

<u>Adjoints</u> Given $(E, d_E) \in \mathrm{hmf}(\mathbb{C}[\underline{x}], W)$ we can define

$$\left(E^{\vee} = \mathrm{Hom}_{\mathbb{C}[x]}(E, \mathbb{C}[x]) , \; d_{E^{\vee}}(\alpha) = (-1)^{|\alpha|} \alpha \circ d_E \right)$$
$$\cap$$
$$\mathrm{hmf}(\mathbb{C}[\underline{x}], -W)$$

Given $X \in \mathcal{LG}\big((\mathbb{C}[x_1, \ldots, x_n], W), (\mathbb{C}[y_1, \ldots, y_m], V) \big)$ we have therefore $X^{\vee} \in \mathcal{LG}(V, W)$ and there are adjunctions in $\mathcal{LG}$

$${}^{\dagger}X = X^{\vee}[m] \dashv X \dashv X^{\vee}[n] = X^{\dagger}$$



<u>Def^n</u> $\mathcal{LG}^{\text{even}}$, $\mathcal{LG}^{\text{odd}}$ are respectively the full sub-bicategories with objects $(\mathbb{C}[x_1, \ldots, x_n], W)$ with n even (resp. n odd or zero).

<u>Theorem</u> $\mathcal{LG}$ is graded pivotal ($\mathcal{LG}^{\text{even}}$, $\mathcal{LG}^{\text{odd}}$ are pivotal in the sense described earlier).

$\therefore$ $\mathcal{LG}^{\text{even}}$, $\mathcal{LG}^{\text{odd}}$ have the property $\circledast$ of last lecture (adjoints on both sides exist and are equal, pivotality, ...).

<u>Units and counits</u> For $X : W(\underline{x}) \longrightarrow V(\underline{y})$ with $|\underline{x}| = n$, $|\underline{y}| = m$ even

$$\widetilde{ev}_X : X \underset{\mathbb{C}[x]}{\otimes} X^\vee \longrightarrow \Delta_V \qquad coev_X : \Delta_V \longrightarrow X \underset{\mathbb{C}[x]}{\otimes} X^\vee$$

are defined for $\gamma \in \Delta_V = \bigwedge \bigoplus_{i=1}^{m} (\mathbb{C}[\underline{x}]^e \theta_i)$ with $\gamma \wedge \theta_B = (-1)^s \theta_1 \cdots \theta_m$ with $B = \{ b_1 < \cdots < b_\ell \}$ by, with $\{e_i\}$ a homogeneous basis of X

$$coev_X(\gamma) = \sum_{i,j} (-1)^{\binom{\ell+1}{2} + s} \underbrace{\{ \partial^y_{[b_1]} d_X \cdots \partial^y_{[b_\ell]} d_X \}_{ij}}_{\mathbb{C}[y] \otimes \mathbb{C}[x] \otimes \mathbb{C}[y]} e_i \otimes e_j^*$$

and for $g \in \mathbb{C}[\underline{x}]$ by

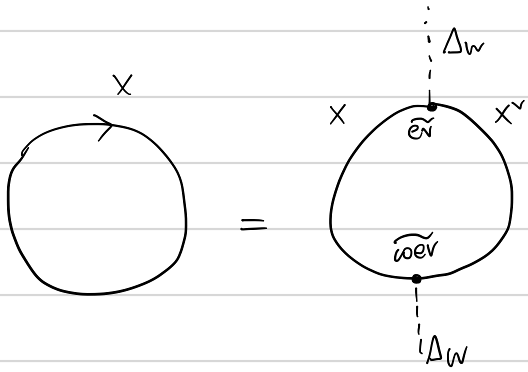
$$\widetilde{ev}_X (g e_j \otimes e_i^*) = \sum_{\ell \geq 0} \sum_{i_1 < \cdots < i_\ell} (-1)^{\ell + |e_j|} \quad \theta_{i_1} \cdots \theta_{i_\ell} \, .$$

$$\operatorname{Res}_{\mathbb{C}[\underline{x},\underline{y}] / \mathbb{C}[\underline{y}]} \left(\frac{ \{ \partial^y_{[i_\ell]} d_X \cdots \partial^y_{[i_1]} d_X \, \partial_{x_1} d_X \cdots \partial_{x_n} d_X \}_{ij} \, g \, dx_1 \cdots dx_n }{ \partial_{x_1} W \cdots \partial_{x_n} W } \right)$$

⑤

<u>Example</u> Let m, n be even, $X : W \longrightarrow V$ so ${}^{\dagger}X = X^{\dagger} = X^{\vee}$

$$J_W = \mathbb{C}[x] / (\partial_{x_1} W, \dots, \partial_{x_n} W)$$

$\dim_r(X) =$ 

$\in \mathrm{End}_{\mathcal{LG}(W,W)}(\Delta_W) \cong J_W$

$$1 \longmapsto \sum_{i,j} (-1)^{\binom{m+1}{2}} \underbrace{\{ \partial^y_{[1]} d_x \cdots \partial^y_{[m]} d_x \}}_{ij} e_i \otimes e_j^*$$

$$\longmapsto \sum_{i,j} (-1)^{\binom{m+1}{2} + |e_j|} \operatorname{Res}_{\mathbb{C}[x,y]/\mathbb{C}[x]} \left(\frac{\{\partial_{x_1} d_x \cdots \partial_{x_n} d_x\}_{ji} \{\cdots\}_{ij} \, dy_1 \cdots dy_m}{\partial_{y_1} V \cdots \partial_{y_m} V} \right) + \theta \text{ terms}$$

$$= (-1)^{\binom{m+1}{2}} \operatorname{Res} \left(\frac{\operatorname{str}(\partial_{x_1} d_x \cdots \partial_{y_1} d_x \cdots) \, \underline{dy}}{\partial_{y_1} V \cdots \partial_{y_m} V} \right) + \theta \text{ terms}$$

It follow that $\dim_r(X) \simeq \operatorname{Res}(\cdots) \cdot 1_\Delta$.

<u>Remark</u> $\mathrm{Hom}_{\mathbb{C}[x]^e}(\Delta_W, \Delta_W) \xrightarrow[\text{qis}]{\pi \circ -} \mathrm{Hom}_{\mathbb{C}[x]^e}(\Delta_W, \mathbb{C}[x])$

$$\cong$$

$$\left(\bigwedge \left(\bigoplus_i \mathbb{C}[x]^e \theta_i^* \right), \ \sum_i \partial_{x_i} W \theta_i^* \right)$$

$$\downarrow \text{qis}$$

$$J_W \cdot 1$$

Graded version

- A *quasi-homogeneous* potential is a potential $W \in \mathbb{C}[x_1, \dots, x_n]$ together with $|x_i| \in \mathbb{Q}_{>0}$ such that $|W| = 2$.

Def^n The bicategory $\mathcal{LG}^{gr}$ has

- objects are quasi-homogeneous potentials, plus zero.

- morphisms a graded MF of quasi-homogeneous $(\mathbb{C}[\underline{x}], W)$ is a MF (X, d_X) of W such that $X^i = \bigoplus_{a \in \mathbb{Q}} X^i_a$ is $\mathbb{Q}$-graded free module, so that X is a $\mathbb{Z}_2 \times \mathbb{Q}$-graded module, and s.t. d_X has bidegree $(1,1)$.

$$\mathcal{LG}^{gr}\big((\mathbb{C}[\underline{x}], W), (\mathbb{C}[\underline{y}], V) \big) := hmf^{gr}(\mathbb{C}[\underline{x}, \underline{y}], V - W)^{\omega}.$$

actually redundant now

- composition as before, giving for $X : (\mathbb{C}[\underline{x}], W) \longrightarrow (\mathbb{C}[\underline{y}], V)$ and $Y : (\mathbb{C}[\underline{y}], V) \longrightarrow (\mathbb{C}[\underline{z}], U)$ a $\mathbb{Z}_2 \times \mathbb{Q}$-graded MF

$$\left(Y \otimes_{\mathbb{C}[\underline{y}]} X, \ d_Y \otimes 1 + 1 \otimes d_X \right)$$

induced $\mathbb{Q}$-grading from grading on Y, X

and this induces a $\mathbb{Q}$-grading on $Y * X$.

- units $\Delta_W \in hmf^{gr}(\mathbb{C}[\underline{x}] \otimes \mathbb{C}[\underline{x}], W \otimes 1 - 1 \otimes W)^{\omega}$ is $\mathbb{Z}_2 \times \mathbb{Q}$-graded with $|\theta_i| = (1, |x_i| - 1)$. The unitors in $\mathcal{LG}$ are clearly bidegree $(0,0)$ so work in $\mathcal{LG}^{gr}$.

see also Ballard-Favero-Katzarkov

Proposition $\mathcal{LG}^{gr}$ is a bicategory

Def^n $G_W = \langle \{ |x_i| \mid 1 \le i \le n \} \rangle \subseteq \mathbb{Q}$, $G_0 := \mathbb{Z} \subseteq \mathbb{Q}$.

so $R\{\lambda\}$ has 1 in deg λ i.e. $R(-\lambda)$

Def^n Let R be a $\mathbb{Q}$-graded ring, $R\{\lambda\}_i = R_{i-\lambda}$ and M a free $\mathbb{Q}$-graded R-module, write $M = \bigoplus_{\lambda \in \mathbb{Q}} R\{\lambda\}^{\oplus m_\lambda}$, and $\mathbb{Q}\text{-spec}(M) = \{ \lambda \in \mathbb{Q} \mid m_\lambda \neq 0 \}$.

Def^n For $a \in \mathbb{Q}$ let
$$\mathcal{LG}^{gr}(W,V)_a := \left\{ X \;\middle|\; \begin{array}{l} \mathbb{Q}\text{-spec}(X^0) \subseteq a + G_{V-W} \\ \mathbb{Q}\text{-spec}(X^1) \subseteq a + 1 + G_{V-W} \end{array} \right\}.$$

Lemma As $\mathbb{C}$-linear categories
$$\mathcal{LG}^{gr}(W,V) = \bigoplus_{a \in \mathbb{Q}/G_{V-W}} \mathcal{LG}^{gr}(W,V)_a$$
and the grading shift defines an equivalence
$$(-)(\lambda) : \mathcal{LG}^{gr}(W,V)_a \longrightarrow \mathcal{LG}^{gr}(W,V)_{a-\lambda}$$

$\vdash$
$$\begin{array}{cc} & R(-\lambda) \\ R(-\mu) & \begin{pmatrix} & \vdots \\ \cdots & f \end{pmatrix} \end{array}$$
$f \in R(\lambda-\mu)_1 = R_{\lambda-\mu+1}$

should have $\lambda - \mu + 1 \in G_W$

$\therefore -\mu \in -\lambda - 1 + G_W$

$\mu \in \lambda + 1 + G_W$

Note $\Delta_W \in \mathcal{LG}^{gr}(W,W)_0$ as it involves free modules $\mathbb{C}[x]^e \theta_i = \mathbb{C}[x]^e(1 - |x_i|) = \mathbb{C}[x]^e\{|x_i| - 1\}$.

Remark
$$X^0 \underset{\deg(1,1)}{\overset{\deg(1,1)}{\rightleftarrows}} X^1 \quad \overset{1:1}{\longleftrightarrow} \quad X^0 \underset{\deg(1,2)}{\overset{\deg(1,0)}{\rightleftarrows}} X^1(1)$$
since $2 \in G_W$, $[1] = [-1]$.

(i.e. Kajiura-Saito-Takahashi)

$$\Sigma(X^0 \rightleftarrows X^1) = X^1(1) \rightleftarrows X^0(1) \longleftrightarrow \text{shift } T(X^0 \rightleftarrows X^1(1)) = X^1(1) \rightleftarrows X^0(2)$$

$$\Rightarrow \left(\mathcal{LG}^{gr}(W,V)_a, \ \Sigma = [1](1) \right) \text{ is triangulated } (\mathbb{Z}\text{-graded})$$

Example $\mathcal{LG}^{gr}(0,0)_0 \cong$ homotopy category of $\mathbb{Z}$-graded cpxs $/\mathbb{C}$.

Def^n The central charge of $W \in \mathcal{LG}^{gr}$ is $\hat{c}(W) = \sum_i (1 - |x_i|)$.

Adjoints Given $X \in \mathcal{LG}^{gr}(W, V)_a$ the coevaluation in $\mathcal{LG}$ is

$$\mathrm{coev}_X : \Delta_V \longrightarrow X \otimes_{\mathbb{C}[x]} X^\vee$$

$$\mathrm{coev}_X(\gamma) = \sum_{i,j} (-1)^{\binom{\ell+1}{2} + s} \{ \underbrace{\partial^y_{[b_1]} d_x \cdots \partial^y_{[b_\ell]} d_x}_{\mathbb{Z}\text{-degree } \sum_i (1-|x_{b_i}|)} \}_{ij} \, e_i \otimes e_j^*$$

$\Rightarrow |e_i| - |e_j| = \sum_i (1 - |x_{b_i}|)$

$$\gamma \wedge \theta_B = (-1)^s \theta_1 \cdots \theta_m$$

$|\theta_i| = |y_i| - 1 \quad |\theta_1 \cdots \theta_m| = -\hat{c}(V)$

$\therefore |\gamma| = -\hat{c}(V) - \sum_i (|x_{b_i}| - 1)$

$$\therefore |\mathrm{coev}| = |e_i| - |e_j| - |\gamma| = \sum_i (1 - |x_{b_i}|) + \hat{c}(V) + \sum_i (|x_{b_i}| - 1)$$
$$= \hat{c}(V).$$

$\Longrightarrow$ coev_X is a degree zero map $\Delta_V \longrightarrow X \otimes_{\mathbb{C}[x]} X^\vee(\hat{c}(V))$

Similarly one shows the other units and counits are homogeneous, so in $\mathcal{LG}^{gr}$

$$X^\vee[n](\hat{c}(V)) \longrightarrow X \longrightarrow X^\vee[m](\hat{c}(W)) \qquad |\underline{x}| = n, |\underline{y}| = m$$

Def^n For $\lambda \in \mathbb{Q}$ let $\mathcal{LG}^{gr}_{c=\lambda} \subseteq \mathcal{LG}^{gr}$ be the full sub-bicategory of W with $\hat{c}(W) = \lambda$.

Proposition $\mathcal{LG}^{gr, \bullet}_{c=\lambda}$ satisfies ⊛ for $\bullet \in \{\text{even}, \text{odd}\}$, $\lambda \in \mathbb{Q}$

has adjoints, is pivotal

From now on $\bullet$ = even for simplicity.

<u>Defⁿ</u> Two singularities $W, V \in \mathcal{LG}^{gr,\,even}_{c=\lambda}$ are (generalised) <u>orbifold equivalent</u> if there exists a 1-morphism $X : W \longrightarrow V$ with invertible dimensions (IVD). Equivalently $Y : V \longrightarrow W$ exists with IVD. We write $W \sim_{GO} V$.

Suppose $W \sim_{GO} V$ via $X : W \longrightarrow V$, then there are adjunctions

$$X^\vee(\lambda) \dashv X \dashv X^\vee(\lambda).$$

By last lecture with $\mathcal{B} = \mathcal{LG}^{gr,\,even}_{c=\lambda}$

- $A := X^\vee(\lambda) \otimes X$ is a separable Frobenius algebra in $\mathcal{B}(W,W)$
- X induces $(W,A) \cong (V, \Delta_V)$ in $\mathcal{B}_{eq}$
- We have for example

$$\begin{tikzcd} & 0 \arrow[dl, dashed] \arrow[dr, dashed] & \\ W \arrow[rr, shift left, "X"] & & V \arrow[ll, shift left, "X^\vee"] \end{tikzcd}$$

$$\begin{aligned} hmf^{gr}(\mathbb{C}[y], V) &= \mathcal{B}(0, V) \\ &= \mathcal{B}_{eq}(0, V) \\ &\cong \mathcal{B}_{eq}(0, (W,A)) \\ &= \text{left } A\text{-modules in } \mathcal{B}(0,W) \\ &= \text{left } A\text{-modules in } hmf^{gr}(\mathbb{C}[x], W) \end{aligned}$$

Moreover the functors are

$$hmf^{gr}(V) \underset{X \otimes_A -}{\overset{X^\vee \otimes -}{\rightleftarrows}} Mod_{hmf^{gr}(W)}(A) \quad (\cong)$$

- For example

$$J_V \cong \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{B}(V,V)}(\Delta_V, \Delta_V(n))$$

$$\cong \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{B}_{eq}(V,V)}(\Delta_V, \Delta_V(n))$$

$$\cong \bigoplus_{n} \mathrm{Hom}_{\mathcal{B}_{eq}((W,A),(W,A))}(A, A(n)) \quad \text{i.e. } \mathrm{End}_{AA}(A)$$

AA-bimodule morphisms $A \longrightarrow A$
as a subalgebra of the graded algebra $\mathrm{End}(A)$

Relation to usual orbifolding (§7.1 of Carqueville-Runkel)

Suppose $(\mathbb{C}[x], W) \in \mathcal{LG}$ with a finite group $G \circlearrowright \mathbb{C}[x]$ s.t. G fixes W.
For $g \in G$ define $\varphi_g : \mathbb{C}[x] \to \mathbb{C}[x]$ and

$$\Delta_g := (\varphi_{g*} \otimes 1)(\Delta_W) \in \mathcal{LG}(W, W).$$

Then for $X : V \longrightarrow W$ we have $\Delta_g \otimes X \cong \varphi_{g*}(X)$.

<u>Lemma</u> $A_G := \bigoplus_{g \in G} \Delta_g$ is a separable Frobenius algebra in $\mathcal{LG}(W, W)$

<u>Proposition</u> $\mathrm{Mod}_{\mathrm{hmf}(W)}(A_G) \cong \mathrm{hmf}(W)^G$ ← G-equivariant MFs in the usual sense.

i.e. the usual orbifolding $\subseteq$ generalised orbifolding, viewed as a theory about the bicategory $\mathcal{B}_{eq}$.

Orbifold equivalence on the level of Jacobi algebras

If $W \sim_{GO} V$ then with $H = \mathrm{End}_{\mathcal{B}(W,W)}(A, A)$ there are $\mathbb{C}$-linear
also Frobenius

$$\begin{tikzcd} & H \arrow[dl, shift right, "\Psi_W"'] \arrow[dr, shift left, "\Psi_V"] & \\ \mathrm{End}(\Delta_W) = J_W \arrow[ur, shift right, "\Phi_W"'] & & J_V = \mathrm{End}(\Delta_V) \arrow[ul, shift left, "\Phi_V"] \end{tikzcd} \qquad A \circlearrowright W \underset{X^\vee}{\overset{X}{\rightleftarrows}} V$$

with Φ_W, Φ_V algebra morphisms, and $\Psi \circ \Phi = 1$, such that

$$\Psi_V \circ \Phi_W = \mathcal{D}_r(X), \quad \Psi_W \circ \Phi_V = \mathcal{D}_l(X).$$

Here Ψ_V is a projector $\mathrm{End}(A) \twoheadrightarrow \mathrm{End}_{AA}(A) \cong J_V$.

Example $W = V^{(A_{2d-1})} = x_1^{2d} + x_2^2$ $\quad |x_1| = \frac{1}{d}$ $\quad |x_2| = 1$ $\quad G_W = \langle \frac{1}{d} \rangle \subseteq \mathbb{Q}$

$V = V^{(D_{d+1})} = y_1^d + y_1 y_2^2$ $\quad |y_1| = \frac{2}{d}$ $\quad |y_2| = 1 - \frac{1}{d}$ $\quad G_V = \begin{cases} \langle \frac{2}{d} \rangle & d \text{ odd} \\ \langle \frac{1}{d} \rangle & d \text{ even} \end{cases}$

Clearly $\hat{c}(W) = 2 - \Sigma_i |x_i| = 2 - (1 + \frac{1}{d}) = 1 - \frac{1}{d} = \hat{c}(V)$,
and $G_{V-W} = G_{W-V} = \langle \frac{1}{d} \rangle$. Then in $\mathcal{LG}^{gr, even}_{c = 1 - \frac{1}{d}} =: \mathcal{B}$ we have

$$\begin{tikzcd} W \arrow[loop left, "A"] \arrow[r, shift left, "X"] & V \arrow[l, shift left, "X^\vee(1-\frac{1}{d})"] \end{tikzcd} \qquad \dim_r(X), \dim_\ell(X) \neq 0.$$

where $A = X^\vee(1 - \frac{1}{d}) \otimes X \cong \Delta_W \oplus \Delta_g[1]$ where $g = -1 \in \mathbb{Z}_2 \curvearrowright \mathbb{C}[x_1, x_2]$
acting by $x_1 \mapsto -x_1, x_2 \mapsto x_2$, where X has degrees

$$X^0 = X^1 = R \oplus R\{-1 + \tfrac{2}{d}\} \quad \text{i.e. } \mathbb{Q}\text{-spec } \{0, -1 + \tfrac{2}{d}\}$$

$\therefore X \in \mathcal{LG}^{gr}(W, V)_0$.

Notice that

$$\begin{tikzcd} & hmf^{gr}(V)_0 \arrow[dl, shift left, "X^\vee \otimes -"] & d \text{ even} \\ hmf^{gr}(W)_0 \arrow[ur, shift left, "X \otimes -"] \arrow[dr, shift left, "X \otimes -"] & & \\ & hmf^{gr}(V)_{\frac{1}{d}} \arrow[ul, shift left, "X^\vee \otimes -"] & d \text{ odd.} \end{tikzcd}$$

Other ADE cases (from Carqueville-Ros Camacho-Runkel '13 and Carqueville-Velez '15)

Recall $V^{(A_{11})} \sim V^{(E_6)}$ so that $W = x_1^{12} + x_2^2$, let $\eta = e^{2\pi i/12}$, for
$\overset{\shortparallel}{W} \quad \overset{\shortparallel}{V}$
$S \subseteq \{0,..,11\}$ define

$$P_S = \begin{pmatrix} 0 & \prod_{i\in S}(y_1 - \eta^i x_1) \\ \prod_{i\notin S}(y_1-\eta^i x_1) & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & y_2 - x_2 \\ y_2 + x_2 & 0 \end{pmatrix}$$

$$\in \mathrm{hmf}(\mathbb{C}[\underline{x},\underline{y}], W(\underline{y}) - W(\underline{x}))$$
$$= \mathcal{LG}(W,W).$$

Then $A := X^\vee \otimes X \cong \Delta_W \oplus P_{\{-3,-2,-1,..,3\}}$. And similarly for E_7, E_8 (in the A-D case, Δ_g from before is $P_{\{d\}}$, and $P_{\{d\}}[1] = P_{S\setminus\{d\}}$). The notion of A-modules is now more complicated, and not well-understood. An A-module structure on E consists of a family of morphisms among E and the $P_{\{a\}} \otimes E$ for various $a \in \mathbb{Z}_d$, according to:

Lemma If $S \subseteq S'$ there is a triangle in $\mathcal{LG}(W,W)$

$$\begin{tikzcd} P_S \arrow[rr] & & P_{S'} \arrow[dl] \\ & P_{S'\setminus S} \arrow[ul, "+1"] & \end{tikzcd}$$