Generalised orbifolding of simple singularities

This talk was originally given at the Geometry at ANU conference in August 2016. The aim is to explain a remarkable theorem of the mathematical physicists Carqueville–Ros Camacho–Runkel, and my own contribution to some of the mathematical inputs to their theorem.

Schematically, the result can be described as an unexpected series of equivalences of triangulated categories associated to isolated hypersurface singularities:

\[ W \in \mathbb{C}[x_1, \ldots, x_n] \quad \Rightarrow \quad \text{triangulated category } \text{hmf}(W) \]

\[ \forall V \in \mathbb{C}[y_1, \ldots, y_m] \quad \Rightarrow \quad \text{" } hmf(V) \text{" } \]

different singularities

The equivalences are not \( hmf(W) \cong hmf(V) \) (this would be too strong to be interesting) but of the form

\[ hmf(V) \cong \text{Mod}_{hmf(W)}(A) \]

Frobenius algebra/monad

The interesting examples so far are pairs \( (V, W) \) of simple singularities (aka ADE singularities), and unimodular singularities, but there are probably many more.

Outline

1. Matrix factorisations i.e. \( hmf(W) \)
2. Frobenius algebra
3. Statement of theorem
4. Sketch of proof (uses work of mine with Carqueville)
Let $R$ be a commutative ring and $W \in R$. A matrix factorisation of $W$ is a $\mathbb{Z}_2$-graded f.g. projective $R$-module $X = X^0 \oplus X^1$ together with an odd $R$-linear $d_X : X \to X$ s.t. $d_X^2 = W \cdot 1_X$. A morphism $\varphi : (X, d_X) \to (Y, d_Y)$ is a degree zero map with $d_Y \varphi = \varphi d_X$.

**Definition** $\text{hmf}(R, W) :=$ matrix factorisations of $W$ with homotopy equivalence classes of morphisms (so $(R \oplus R, (0, W)) \cong 0$).

**Examples**

1. $R = \mathbb{C}[x]$, $W = x^N$ $N \geq 2$, for $1 \leq i \leq N - 1$

   $E_i := \left( R \oplus R, \begin{pmatrix} 0 & x^i \\ x^{-i} & 0 \end{pmatrix} \right) \in \text{hmf}(\mathbb{C}[x], x^N)$

2. $R = \mathbb{C}[x, y]$, $W = y^N - x^N$ $N \geq 2$, $\gamma = e^{2\pi i/N}$ so

   $y^N - x^N = \prod_{0 \leq i \leq N - 1} (y - \gamma^i x)$

   Given $S \subseteq \{0, \ldots, N - 1\}$ we have

   $P_S := \left( R \oplus R, \begin{pmatrix} 0 & \prod_{i \in S} (y - \gamma^i x) \\ \prod_{i \notin S} (y - \gamma^i x) & 0 \end{pmatrix} \right) \in \text{hmf}(\mathbb{C}[x, y], y^N - x^N)$

**Theorem** (Buchweitz, Orlov) There is an equivalence of triangulated categories

$$\text{hmf}(\mathbb{C}[x, \ldots, x_n], W) \cong \mathbb{D}^b(\omega W^{-1}(0)) / \text{Perf}(W^{-1}(0))$$
Proposition: If \( W \in \mathcal{C}[x_1, \ldots, x_n] \) has isolated critical points then

\[
\mathcal{E}_W = \text{hmf}(\mathcal{C}[x] \otimes \mathcal{C}[x], W) \quad \text{Koszul closure / idempotent closure}
\]

is naturally a \( \otimes \)-triangulated category \((\mathcal{E}_W, \ast)\), and \( \text{hmf}(\mathcal{C}[x], W) \)

is an \( \mathcal{E}_W \)-module, i.e. there is an action \( \mathcal{E}_W \times \text{hmf}(W) \to \text{hmf}(W) \)

where the tensor product \( \ast \) is for \( X, Y \in \mathcal{E}_W \)

\[
Y \ast X := \text{finite rank representative of } \left( \bigotimes_{\sigma(x)} X, d_Y \otimes 1 + 1 \otimes d_X \right) \text{ in } \mathcal{E}_W.
\]

and the action of \( X \) on \( E \in \text{hmf}(\mathcal{C}[x], W) \) is

\[
Y \ast E := \text{finite rank representative of } \left( \bigotimes_{\sigma(y)} E, d_Y \otimes 1 + 1 \otimes d_E \right) \text{ in } \text{hmf}(W).
\]

Example: \( \mathcal{E}_x', \mathcal{E}_x' = \text{hmf}(\mathcal{C}[x,y], y^n - x^n) \) is monoidal, with \( 0 \leq \lambda \leq N - 2 \)

\[
P_a : \lambda := P_{\{a, a+1, \ldots, a+\lambda\}} \in \mathcal{E}_{x^N}
\]

Example: \( P_a : \lambda \ast P_{b, 0} = P_{a+b : \lambda} \)

\[
P_a : \lambda \ast P_{b : \mu} = \bigoplus_{\nu = |\lambda - \mu|} P_{a+b - \frac{1}{2}(\mu+\lambda)-\nu} : \nu
\]

by steps of 2 \[\text{[Brunner–Roggenkamp '07]}\]

related to fusion / \( \text{su}(2)_{N-2} \)
2) Frobenius algebras

Let \((\mathcal{C}, \otimes, I)\) be a monoidal category. A \textit{Frobenius algebra} in \(\mathcal{C}\) is an object \(A \in \text{ob}(\mathcal{C})\) equipped as an

- associative, unital algebra, \(\mu : A \otimes A \to A, \eta : A \to I\)
- coassociative, counital coalgebra, \(\Delta : A \to A \otimes A, \varepsilon : I \to A\)

such that the Frobenius identity holds:

\[
\begin{array}{c}
\begin{aligned}
\mu & \circ (1_A \otimes \mu) \circ (\Delta \otimes 1_A) = \mu \\
& = \Delta \circ \mu = (\mu \otimes 1_A) \circ (1_A \otimes \Delta)
\end{aligned}
\end{array}
\]

A \textit{Frobenius algebra} is \textit{separable} if

\[
\begin{array}{c}
\begin{aligned}
(1_A \otimes \mu) \circ (\Delta \otimes 1_A) = \mu \circ (\mu \otimes 1_A) \\
\end{aligned}
\end{array}
\]

If there is an action \(\mathcal{C} \times J \to J, A \otimes - : J \to J\) is a monad, and \(\text{Mod}_J(A)\) denotes modules over this monad.

\textbf{Theorem (Balmer)} If \(\mathcal{C}\) is a \(\otimes\)-triangulated category acting on a triangulated category \(J\), and \(A\) is a separable algebra in \(\mathcal{C}\), then \(\text{Mod}_J(A)\) is naturally triangulated (some caveats).
3. From now on $W \in \mathbb{C}[x_1, \ldots, x_n]$, $V \in \mathbb{C}[y_1, \ldots, y_m]$ have isolated cut points.

**Definition**: $V$ is a weak generalised orbifold (WGO) of $W$, denoted $W \rightarrow_{\text{wao}} V$, if there is a separable Frobenius algebra $A \in \text{End}_W$ and an equivalence of triangulated categories

$$\text{Mod}_{\text{hmfr}(w)}(A) \cong \text{hmfr}(V) \quad (\text{note } \text{hmfr}(W) \supset A)$$

**Example**

1. $W \rightarrow_{\text{wao}} W$, $A = \Delta_W \in \text{End}_W$ which is the monoidal unit

2. $W \rightarrow_{\text{wao}} W + u^2 + v^2$, $A = \Delta_W \otimes_{\text{C}} \text{Cliff}(u^2 + v^2)$

(a form of Knörrer periodicity)

**Theorem** (Carqueville - Ros Camacho - Runkel '13) With the notation

<table>
<thead>
<tr>
<th>$V^{(\text{ADE})}$</th>
<th>$c$</th>
<th>$d$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V^{(\text{A}_{d-1})}$</td>
<td>$x_1^d + x_2^2$</td>
<td>$c = 3 - 3 \cdot \frac{2}{d}$</td>
</tr>
<tr>
<td>$V^{(\text{D}_{d+1})}$</td>
<td>$x_1^d + x_1 x_2^2$</td>
<td>$c = 3 - 3 \cdot \frac{2}{2d}$</td>
</tr>
<tr>
<td>$V^{(\text{E}_6)}$</td>
<td>$x_1^3 + x_2^4$</td>
<td>$c = 3 - 3 \cdot \frac{2}{12}$</td>
</tr>
<tr>
<td>$V^{(\text{E}_7)}$</td>
<td>$x_1^3 + x_1 x_2^3$</td>
<td>$c = 3 - 3 \cdot \frac{2}{18}$</td>
</tr>
<tr>
<td>$V^{(\text{E}_8)}$</td>
<td>$x_1^3 + x_2^5$</td>
<td>$c = 3 - 3 \cdot \frac{2}{30}$</td>
</tr>
</tbody>
</table>

we have

\begin{align*}
(V^{(\text{A}_{d-1})}) & \rightarrow_{\text{wao}} (V^{(\text{D}_{d+1})}) \\
(V^{(\text{A}_n)}) & \rightarrow_{\text{wao}} (V^{(\text{E}_6)}) \\
(V^{(\text{A}_n)}) & \rightarrow_{\text{wao}} (V^{(\text{E}_7)}) \\
(V^{(\text{A}_2)}) & \rightarrow_{\text{wao}} (V^{(\text{E}_8)})
\end{align*}

i.e. "\text{ADE from } A"

(true for ADE in any dimension not just curves)
Note. By a theorem of Kajiura-Saito-Takahashi for $V$ an ADE singularity

$$\text{hm}f_{W}^{\mathfrak{g}}(\mathbb{C}[x,t], V + t^2) \cong D^b(\text{rep. } C\overline{Q})$$

where $Q$ is the corresponding Dynkin quiver. The first of the above wGO relations was known (Reiten - Riedtmann '85) and corresponds to a "folding"

But the $A \to E$ relations don't seem to arise some group actions on quiver in this straightforward way.
Sketch of proof (assume m, n even)

Prop (Carqueville-M ’12) Any $X \in \text{hm}f(C[x,y], V(z) - W(z))$
has a dual $X^\vee \in \text{hm}f(C[x,y], W(z) - V(y))$ which is an adjoint (on both sides)
in an appropriate bicategory

\[
\begin{array}{cc}
X & \xrightarrow{\text{unit}} X^* \xrightarrow{\text{count}} X^* \\
\text{W} & \xrightarrow{\text{co-unit}} \text{V}
\end{array}
\]

\[
\begin{array}{cc}
X^* & \xrightarrow{\text{unit}} X \xrightarrow{\text{count}} X^* \\
\text{W} & \xrightarrow{\text{co-unit}} \text{V}
\end{array}
\]

This leads us morphisms in $\mathcal{E}^W, \mathcal{E}^V$ resp.

\[
\begin{align*}
\text{qdim}_{\text{left}}(X) & := \Delta_W^{-1} X^* X \\
\text{qdim}_{\text{right}}(X) & := \Delta_V^{-1} X^* X
\end{align*}
\]

Prop If both qdim’s of $X$ are scalar multiples of $1_{A}$, then $A := X^\vee \ast X$
is a separable Frobenius alg. in $\mathcal{E}^W$ and $\text{Mod}_{\text{hm}f(W)}(A) \cong \text{hm}f(V)$,that is, $W \longrightarrow \text{wao} V$. (a version of the Barr-Beck theorem)

Def If $X$ as in the proposition exists we say $W, V$ are orbifold equivalent

$V \sim_{\text{co}} W$. This is an equivalence rel $\mathcal{M}$ stronger than $\longrightarrow \text{wao}$.

One actually proves the ADE singularities are orbifold equivalent, e.g. $\sqrt{(A_d-1)} \sim_{\text{co}} \sqrt{(D_{4d+4})}$

Note For a grading $|x_i| \in \mathbb{Q}$ i.t. $|W| = 2$, the central charge is $\hat{c}(W) = \sum_i (1 - |x_i|)$

Lemma $V \sim_{\text{co}} W \implies m \equiv n \pmod{2}$ and $\hat{c}(W) = \hat{c}(V)$.

\[
\begin{align*}
V(D_{4d+4}) & = x_1^{d_1} x_2^{d_2} \quad |x_1| = \frac{2}{d_1}, \quad |x_2| = 1 - \frac{1}{d_1}, \quad c = 1 - \frac{1}{d_1}, \quad \text{the same as} \\
V(A_{2d-1}) & = y_1^{2d} + y_2^{2d} \quad |y_1| = \frac{1}{d}, \quad |y_2| = 1.
\end{align*}
\]
Theorem (Carqueville-M '12) With $X: W \to V$ as above,

$$qdim_{\mathbb{C}}(X) = \pm \text{Res}_{x \in \mathbb{C}[x]} \left( \prod_{i=1}^{\text{str}(M)} \frac{\partial x_{\text{idx}} \cdots \partial x_{\text{idx}} \partial y_{\text{idx}} \cdots \partial y_{\text{idx}}}{\partial y_{i} \cdots \partial y_{m} V} \right) \cdot 1_{\Delta}$$

$$\text{str}(M) = \sum_{i} (-1)^{i+1} M_{ii}$$

and similarly for $qdim_{\mathbb{R}}(X)$.

Finally: Carqueville - Ros Camacho - Runkel prove their theorem by searching the space of matrices $d_{x}^{2}$ over $\mathbb{C}[x,y]$ with (i) $dx_{x} = V - W$ and (ii) $qdim_{\mathbb{C}}(X) \in \mathbb{C}^{\ast}$, $qdim_{\mathbb{R}}(X) \in \mathbb{C}^{\ast}$ in a clever way, and finding an explicit $d_{x}$ in each case.

Notes

- Of ADE pain

  - In all cases the Frobenius algebra $A := X \ast X \in Ew$ has an underlying object a direct sum of $P_{S}$ matrix factorisations for some $S$ (Carqueville)

  - Conjecture Strangely dual unimodular exceptional singularities are orbifold equivalent (there are four nontrivial cases, as 6 out of 14 are self-dual).

Known:

- $Q_{10} : x^{4} + y^{3} + x^{2} \sim Q_{0}$
- $E_{6} : x^{4} + y^{3} + z^{2}$ (Ros-Camacho, Newton '15)

- Exceptional unimodular sing. of same weight ($a_{4}, q_{2}, a_{3}, h$) are GO-equivalent (Ros-Camacho, Newton '16).

  $$|\chi_{i}| = \frac{2a_{i}}{h} \quad cw = \frac{h + 2}{h}$$

Total 64
Q1/ What is the geometric origin of orbifold equivalences?

Q2/ Is there a better way of generating examples?
Appendix A
Strange duality, from Ebeling “Strange duality, minor symmetry and the Leech lattice” ’98
Appendix B  ADE orbifolding defects $X$

- $\sqrt{(A_{d-1})} \sim \sqrt{(D_{12} + 1)}$ \quad \text{rank } X = 2 \quad \text{(i.e. } d_X^1, d_X^0 \text{ are } 2 \times 2 \text{ matrices)}$
- $\sqrt{(A_{11})} \sim \sqrt{(E_6)}$ \quad \text{rank } X = 2$
- $\sqrt{(A_{10})} \sim \sqrt{(E_7)}$ \quad \text{rank } X = 2$
- $\sqrt{(A_{29})} \sim \sqrt{(E_8)}$ \quad \text{rank } X = 4$
References

• Carqueville, Ros-Camacho, Runkel “Oblifold equivalent potentials” arXiv: 1311.3354.

• Carqueville, Runkel “Oblifold completion of defect bicategories” arXiv: 1210.6363.


• Davydo, Ros Camacho, Runkel “N=2 minimal conformal field theories and matrix bifactorisations of xd” arXiv: 1409.2144.


• Newton, Ros Camacho “Strangely dual obblifold equivalence II”