The aim of today’s lecture is to compute (as far as we are able) the $A_\infty$-minimal model of the DG-category $mf(R,W)$ where $k$ is a $\mathbb{Q}$-algebra, $R = k[x_1,\ldots,x_n]$ and $W \in R$ is a potential.

**Notation:** $C = mf(W)$, $I = (x_1W,\ldots,x_nW)$, $\hat{R}$ is I-adic completion, $Jac_W = R/I$

$$\zeta^* : Jac_W \otimes_k k[[\underline{t}]] \rightarrow \hat{R}$$

$$\nabla : \hat{R} \rightarrow \hat{R} \otimes_{k[[\underline{t}]]} \bigoplus_{n \geq 1} k[[\underline{t}]]/k = \sum_i \frac{\partial}{\partial t_i} dt_i$$

**Def** An $A_\infty$-category $\mathcal{A}$ over $k$ consists of a class $ob(\mathcal{A})$ of objects, for each pair $a, b \in ob(\mathcal{A})$ a $\mathbb{Z}_2$-graded $k$-module $\mathcal{A}(a,b)$, and for $n \geq 1$ and a sequence $a_0,\ldots,a_n \in ob(\mathcal{A})$ a degree $2-n$ $k$-linear map

$$m_{a_0,\ldots,a_n} : \mathcal{A}(a_{n-1},a_n) \otimes_k \cdots \otimes_k \mathcal{A}(a_1,a_2) \otimes_k \mathcal{A}(a_0,a_1) \rightarrow \mathcal{A}(a_0,a_n)$$

subject to the $A_\infty$-constraints for $n \geq 1$ given by

$$\sum_{\substack{i+j+n \geq 1 \leq i+j \leq n \atop i+j > 1}} (-1)^{i+j+n} m_{n-j+1}(x_n \otimes \cdots \otimes x_{i+j+1} \otimes m_j(x_{i+j} \otimes \cdots \otimes x_{i+1}) \otimes x_i \otimes \cdots \otimes x_0) = 0.$$

**Remark**

(i) $m_{a_0,a_n} \subseteq \mathcal{A}(a_0,a_1)$, $m_{a_0,a_0} = 0$. Write $m_1 = \{m_{a_0,a_0}\}_{a_0,a_0}$.

(ii) $H_{m_1}^k(\mathcal{A})$ is a $k$-linear category

(iii) Any DG category is an $A_\infty$-category.
**Def.** Let $A$ be an $A\infty$-category. Then a quasi-isomorphism of $A\infty$-categories $F: B \longrightarrow A$ is called a

(i) **minimal model** if $m_1^B = 0$.

(ii) **finite model** if $B(a,b)$ is a f.g. projective $k$-module for all $a, b \in \text{obj}(B)$, and each map

$$F_{a,a_0}: B(a,a_0) \longrightarrow A(a,a_0)$$

is a $k$-linear homotopy equivalence.

**Motivation** Why consider $A\infty$-categories? There are various reasons, but here are two:

1. Given a DG-category $A$ over a field $k$, it is common that each $A(a,b)$ is an $\infty$-dim vector space but $H^*A(a,b)$ is finite. The minimal model

$$\mathcal{B} = (H^*A, \{m_k^B\}_{k \geq 2}) \xrightarrow{q_i} (A, m_1, m_2) = A$$

therefore gives a finite-dimensional model of $A$, which is "lossless" in the sense that $\mathcal{B}$, $A$ are Morita equivalent. This is applied, for instance, to study deformations of objects of $A$, or $A$ itself.

2. In examples, the calculation of $m_k^B$'s tend to distill the "fundamental homological invariants" of objects of $A$, i.e. the Atiyah class in the case of matrix factorisations.
Recall from last lecture:

**Lemma (Completion comparison)** The DG-functor $C \to C \otimes R \hat{R}$ is a $k$-linear homotopy equivalence (i.e. all induced $C(X,Y) \to C(X,Y) \otimes R \hat{R}$ are h.e.)

**Def.** We define the DG-category

$$
\mathcal{C}_\Theta = \wedge(k_{0_1} \otimes \ldots \otimes k_{0_n}) \otimes_k C \otimes_k \hat{R},
$$

where $\wedge(k_{0_i} \otimes)$ and $\hat{R}$ are viewed as DG-algebras with zero differential.

**Theorem (M)** There is an SDR $\pi$ for any $X,Y \in \mathcal{C}$

$$
H_\infty \subset \mathcal{C}_\Theta(X,Y) \xrightarrow{\pi} C(X,Y) \otimes R \text{Jac}_w
$$

and hence a finite model of the DG-category $\mathcal{C}_\Theta$

$$
\left(\mathcal{C} \otimes \text{Jac}_w, \{m_k\}_{k \geq 1}\right) \xrightarrow{\text{qis}} \mathcal{C}_\Theta.
$$

induced higher products

Next we sketch the proof, highlighting the role of Atiyah classes, before explaining how this may be used to give a minimal model of $C = mf(W)$.

**Remark.** As mentioned in Lecture 2, both Seidel and Efimov have used the SDR from Lecture 1 to compute minimal models of $\text{End}_R(k_{stb})$. However this SDR does not apply to all of $C$, so cannot be used to compute minimal models of e.g. $\text{End}_R(X)$. 
Def. Set $\mathcal{E}(X,Y) := \mathcal{E}(X,Y) \otimes \hat{R}$. There is a $k$-linear flat connection

$$\nabla : \mathcal{E}(X,Y) \to \mathcal{E}(X,Y) \otimes k[\xi] \otimes \mathcal{E}(X,Y)$$

and the relative Atiyah class of the pair $X, Y$ is the morphism of $k[\xi]$-complexes

$$[\nabla, \partial \xi(X,Y)] : \mathcal{E}(X,Y) \to \mathcal{E}(X,Y) \otimes k[\xi] \otimes \mathcal{E}(X,Y).$$

Sketch of proof. Fix $X, Y \in \mathcal{E} = mf(R, W)$. Then we have an SDR over $k$ between the top and bottom rows

$$\bigcup H = \left( \bigwedge (\Theta \otimes k \Theta_i) \otimes k \mathcal{E}(X,Y) \otimes \hat{R}, \partial \xi(X,Y) \right) \quad \mathcal{S} = \sum_i \lambda_i \Theta_i^*$$

$$\equiv \exp(-\xi) \quad \exp(\xi)$$

$$\left( \bigwedge (\Theta \otimes k \Theta_i) \otimes k \mathcal{E}(X,Y) \otimes \hat{R}, \partial \xi(X,Y) + \partial k \right) \quad \partial k = \sum_i \partial \xi_i W \Theta_i^*$$

$$\mathcal{C} \otimes \text{Jac} \left( \mathcal{E}(X,Y) \otimes k \text{Jac}, \partial \xi(X,Y) \otimes 1 \right)$$

e.g. $\{\exp(d) \partial \xi\} \circ \{\pi \exp(-d)\} = 1 - [\partial \xi(X,Y), H \partial \xi]$. The Clifford representation on $\mathcal{C} \otimes \text{Jac}$ induced by the $\Theta_i, \Theta_i^*$ acting on $\mathcal{C}$ was worked out explicitly in [M] in terms of Atiyah classes.
Here the nontrivial components are (roughly speaking) both polynomials in Atiyah classes,

\[ H_\infty = \sum_{m \geq 0} (-1)^m \left[ \nabla, d \mathcal{E} \right]^m \nabla \quad \xi_\infty = \sum_{m \geq 0} (-1)^m \left[ \nabla, d \mathcal{E} \right]^m \xi \]

**Diagram:**

\[ \text{DG} \xrightarrow{\text{finite/minimal model}} \mathcal{C} \xrightarrow{\text{complete minimal model}} \mathcal{C} \xrightarrow{\text{injective (!)}} \mathcal{C} \xrightarrow{\text{minimal model}} \mathcal{C} \]

\[ \text{End}_R(\mathbf{p}^{\text{stab}}) \xrightarrow{\text{can}} (H^* \text{End}_R(\mathbf{p}^{\text{stab}}), \{m_k\}_{k \geq 2}) \]

\[ \mathcal{C} \xrightarrow{\text{can}} (H^* \mathcal{G}, \{m_k\}_{k \geq 2}) \]

\[ \mathcal{C} \xrightarrow{h.c.} \mathcal{C} \xrightarrow{\text{minimal model}} \mathcal{C} \]

\[ \text{complete finite model} \]

**Summary:** We extend \( \mathcal{C} \) to \( \mathcal{C}_0 \), take a minimal model, and project back:

\[ (H^*(\mathcal{C} \otimes \text{Jac} w), \{m_k\}_{k \geq 1}) \xrightarrow{\text{can}} \Lambda (k \otimes \cdots \otimes k \otimes \mathcal{C}_0) \otimes_k (H^* \mathcal{G}, \{m_k\}_{k \geq 2}) \]

\[ \bigcup_{i, \hat{i}} \Theta_i, \Theta_{\hat{i}} \overset{\text{induced action}}{\longrightarrow} \bigcup_{i, \hat{i}} \Theta_i, \Theta_{\hat{i}} \overset{\text{Ker}(\Theta_{\hat{i}}^*)}{\longrightarrow} \bigcup_{i} \text{Ker}(\Theta_i^*) \]

"complete" = we will give the Feynman rules in terms of \( W \)
Feynman rules (End_{k^{\text{stab}}} case, W has no quadratic terms)

The minimal model is \( \left( \bigwedge (\Theta_i = \epsilon_i, k \xi_i), \{m_k\}_{k \geq 2} \right) \) where

\[ m_k : \bigwedge (k \xi) \otimes^k \longrightarrow \bigwedge (k \xi) \]

is computed by a sum over connected planar rooted trees with \( k \)-inputs and internal vertices of degree \( \Xi \), with trees decorated by Feynman diagrams built from the following local vertices, associated to a choice of factorisation

\[ W = \sum_i \chi_i W^i \quad W^i = \sum_{\sigma \in \text{Perm}} W^{i \sigma} x^\sigma \]

\[ A\text{-type} \]

(on inputs and internal edges, any number)

\[ B\text{-type} \]

(exactly one on each int. edge)

\[ C\text{-type} \]

(only at internal vertex, one "leg" in each incoming branch)

Although it is obscured, this interaction vertex "is" the Atiyah class of \( \text{End}(k^{\text{stab}}) \).
Example For \( W = x^3 \in k[x] \), a char.0 field, we have \( W = x \cdot x^2 \), so there is only one kind of \( A \)-type

An example of a Feynman diagram constructed from these local interactions is

Scalar factors The operator \( \frac{1}{[\nabla, d \sigma]^{-1}} \) contributes scalar factor to all of these diagrams, of the form

\[
\sum_{\gamma \in S_n} \frac{1}{(a + a_{\delta(1)})(a + a_{601} + a_{6(2)}) \cdots (a + a_{601} + \cdots + a_{6(n)})}
\]

but while important, we will not describe these factors in these lectures.
Feynman rules (General case, $\text{End}_R(\mathbb{X})$)

For simplicity we will assume $\mathbb{X}$ is of Koszul type

$$\mathbb{X} = \{ f, g \} = \bigwedge (k \mathbb{S}_1 \oplus \ldots \oplus k \mathbb{S}_e), \quad \Sigma_i \mathbb{S}_i \mathbb{S}_i^* + \Sigma_i g_i \mathbb{S}_i^*.$$

The $B, C$-type Feynman rules are as before. To describe the $A$-type interactions we choose a $k$-basis $\text{Jac}_W = \bigoplus_{i=1}^k k \cdot \mathbb{Z}_i$. We need the tensor $T$, discussed in Lecture 1, namely

$$T = (T_i^j) \in \text{Jac}_W^* \otimes_k \text{Jac}_W^* \otimes_k \text{Jac}_W \otimes_k k[[t, \ell]] \quad (\beta \in \mathbb{N})$$

given by the $k$-linear map

$$\text{Jac}_W \otimes_k \text{Jac}_W \xrightarrow{z \otimes z} \hat{R} \otimes_k \hat{R} \xrightarrow{\text{mult}} \hat{R} \xrightarrow{\pi} \text{Jac}_W \otimes_k k[[t, \ell]].$$

Given $f \in \hat{R}$ recall that $f = \sum M z(f_{n}) t^{M} = \sum_{i, \ell} f_{n, \ell} z(\mathbb{S}_{i}) t^{M}$, $f_{n, \ell} \in k$.

With this notation there are two $A$-type interactions, of which we display only one:

**$A$-type**

\[
\begin{array}{c}
\text{coeff} (f_{u}) \cdot_{k} \mathbb{X} \\
\text{coeff} \quad \text{constraint} \quad \epsilon + 1 \ell = d \\
\ell \\
\Omega_{j} \\

\end{array}
\]
Example: For \( W = \frac{1}{5} x^5 \in k[x] \), \( k \) char. 0 field, and \( t = x^4 \),

\[
\chi = \{ x^2, x^3 \} = (\wedge(k^3), x^2 \wedge + \frac{1}{5} x^3 \wedge)
\]

\[
\text{Jac}_W = \bigoplus_{j=0}^3 k z_j, \quad z_j = x^j, \quad (f=x^2, g=x^3)
\]

This diagram contributes to

\[
m_3(\tilde{z} \otimes \tilde{z} \otimes x^3 \tilde{z}) = 1 + \ldots
\]

Again, the \( A^- \) type interaction is a piece of the Atiyah class of \( \text{End}_k(X) \).
Conclusion  We have discussed, for a potential $W \in k[z_1, \ldots, z_n]$, to define connections, for $R = k[z]$, 

$$\nabla: \hat{R} \to \hat{R} \otimes_{k[z]} \Omega^1_{k[z]/k}$$

which "differentiate" in the normal directions $\xi$ to the critical locus. Using these connections we constructed strong deformation retracts, and used them to prove various facts about the DG-category of matrix factorisations, including a description of the Feynman rules in minimal models.

Perhaps the main conceptual insight to be gained from the construction of $A_\infty$-minimal models that we have presented is that it "proves" the following empirical observation:

**Slogan**: Every homological invariant of matrix factorisations is a function of Atiyah classes.

We can justify this as follows. Let $Z(-)$ be a homological invariant:

$$\begin{align*}
mf(W) & \xrightarrow{\varphi} \{H^0\mf(W), \{mk\}_{k>2}\} \\
\xrightarrow{\alpha} & \\
\xrightarrow{Z} & \\
\xrightarrow{U} & \\
\xrightarrow{Z} &
\end{align*}$$

Then it should be invariant under DG quasi-isomorphism. But then $Z(X) = ZF(X)$ and the only way that $X$ enter the $A_\infty$-structure on $F(X)$ is via its Atiyah class, which determine the $A$-type interactions.
References


