The aim of these lectures is to present the Feynman rules governing the $A_{\infty}$-minimal model of the DA-category of matrix factorisations of a potential $W \in k[x_1, \ldots, x_n]$. Let us begin with a rough sketch of this destination, before we get into technical matters necessary to construct the minimal model.

Potential $W \xrightarrow{\text{lossy}}$ DA-category $\text{mf}(W)$

$\xrightarrow{\text{triangulated cat.}} \text{hmf}(W) = H^0 \text{mf}(W)$

$\xrightarrow{\text{lossless}} A_{\infty}$-category finite $/ k$

(topological string theory, LG/CFT)

Example. For $W = x^3 \in k[x]$, $k$ a char.0 field, a relevant Feynman diagram is

As usual, the Feynman diagram depicts a certain pattern of contractions between creation and annihilation operators acting on the $\mathbb{Z}_2$-graded vector space $\mathcal{H}$. For this $W$ there are exactly three kinds of interaction vertices, labelled A, B, C. This diagram computes the only nontrivial contribution to the $A_{\infty}$-structure,
\[ m_3 : \Lambda (k \varepsilon)^{\otimes 3} \rightarrow \Lambda (k \varepsilon), \quad m_3(\varepsilon^{\otimes 3}) = 1. \]

Thus the tuple \( \mathcal{B} = (\Lambda (k \varepsilon), m_2, m_3) \) is an \( A_\infty \)-algebra \( (m_2 \) being the usual product in the exterior algebra) and \( \mathcal{B} \) is the minimal model of the endomorphism \( DA \) of the standard generator of the DG-category \( mf(W) \). In particular \( \text{Perf}_A \cong hf_{\infty}(k[\varepsilon], x^3) \).

Outline of Lectures

1. Connections and contracting homotopies.
2. The DG-category \( mf(W) \) and generators.
3. The \( A_\infty \)-minimal model.

Lecture I

The technical core of the \( A_\infty \)-calculations will be certain connections produced from quasi-regular sequences, which we now review. Let \( R \) be a commutative \( k \)-algebra, for some base ring \( k \).

**Def.** A sequence \( a_1, \ldots, a_n \in R \) is quasi-regular if, writing \( I = (a_1, \ldots, a_n) \), the following morphism of \( R/I \)-algebras

\[ \phi : R/I[t_1, \ldots, t_n] \rightarrow gr R = R/I^2 \otimes I^2/I^3 \otimes \cdots \]

\[ \phi(t_i) = a_i \in I/I^2 \]

is an isomorphism. In particular \( I/I^2 \cong \bigoplus_{i=1}^n R/I . a_i \).
Remark: Set $Y = \text{Spec}(R/I) \hookrightarrow \text{Spec}(R) = X$, then $\mathcal{C}_Y X := \text{Spec}(\text{gr}_I R)$ is a scheme over $Y$ called the normal cone. When $I$ is generated by a quasi-regular sequence, the normal sheaf $(t_0 Y \text{ in } X)$ is a bundle.

$$(I/I^2)^* := \text{Hom}_{R/I}(I/I^2, R/I)$$

is free of rank $n$ on the basis $\{ q_i^* \}_{i}$, and the (total space of the) normal bundle is the relative Spec

$$\text{Spec}_{\mathcal{O}_Y} \left( \text{Sym}(\mathcal{N}_{Y/X}) \right) = \text{Spec} \left( \text{Sym}_{R/I}(I/I^2) \right)$$

$$= \text{Spec} \left( R/I [t_0, \ldots, t_n] \right), \quad t_i = \overline{q}_i$$

$$= \mathcal{C}_Y X,$$

i.e., the normal cone is the normal bundle.

The method of deformation to the normal cone (used in e.g., intersection theory) relates the closed immersion $Y \hookrightarrow X$ by a flat deformation to the inclusion $Y \hookrightarrow \mathcal{C}_Y X$. By the above, in the quasi-regular cone, this latter immersion is just the zero section of the normal bundle. This is the algebraist's analogue of the tubular neighborhood theorem for smooth manifolds: if $Y \subset X$ is a submanifold, a partial tubular neighborhood is a neighborhood $U$ of the zero section of $N_Y X \rightarrow Y$ and an embedding $f : U \rightarrow X$ s.t. $f|_Y = i$ and $f(U)$ is open in $X$:

(smooth manifolds)
Returning to the algebraic case, we do not expect $Y$ to have isomorphic Zariski open neighborhoods in $G_X$ and $X$, but we could ask for the formal neighborhoods to be isomorphic, as in the diagram

$$\begin{array}{c}
C_Y X \leftarrow (C_Y X)_Y \\
\| \| \| \\
\| \| \| \\
Y \leftarrow X \\
X^Y \\
\end{array} \quad (\text{schemes})$$

Now the completion of $C_Y X$ along $Y$ just means passing from $R/I[[t, \ldots, t_n]]$ to $R/I[[t, \ldots, t_n]]$, and completing $X$ along $Y$ means taking the $I$-adic completion $\hat{R}$. So we are asking for a ring isomorphism (in fact a $R[[t]]$-algebra isomorphism)

$$R/I[[t, \ldots, t_n]] \cong \hat{R}.$$ 

**Example** Let $k$ be a field, $R = k[x]$ and $I = (x^d)$ for $d > 1$. By dimension count

$$\left( k[x]/(x^d) \right)[t] \cong \bigoplus_{i > 0} \frac{(x^{di})}{(x^{di+d})}$$

so $a = x^d$ is certainly quasi-regular. The $I$-adic topology is the same as the $(x)$-adic topology, so

$$R/I[[t]] = k[x]/(x^d) \otimes_k k[[t]]$, \quad \hat{R} = k[[x]].$$

There are certainly not isomorphic (one is reduced, the other is not).

The attempt $\otimes$ at an algebraist's "formal" tubular neighborhood is too naive, but there is a useful substitute:
Lemma (Lipman) Suppose $a_1, \ldots, a_n \in R$ quasi-regular, that $R/I$ is a finitely presented $k$-module and that the quotient $\pi: R \to R/I$ has a $k$-linear section $\delta: R/I \to R$. Then there is an induced isomorphism of $k[t_1, \ldots, t_n]$-modules

$$\delta^*: R/I \otimes_k k[[t]] \to \hat{R}.$$ --- “formal tubular neighborhood"

Proof First of all, there is a commutative diagram

$$
\begin{array}{ccc}
R & \xrightarrow{\pi} & R/I \\
\downarrow & & \downarrow \cong \\
\hat{R} & \xrightarrow{\pi} & \hat{R}/I\hat{R}
\end{array}
$$

(note $\hat{I} \subset \hat{R}$ is $I\hat{R}$)

so we do not distinguish $R/I$, $\hat{R}/I\hat{R}$. Since $\hat{R}$ is a $k[[t]]$-algebra ($t_i$ acting as $a_i$) any section $\delta$ induces a $k[[t]]$-linear map $\delta^*$ defined by

$$\delta^*(r \otimes f(t)) = \delta(r) \cdot f(t).$$

To show that $\delta^*$ is an isomorphism we must first show that every $r \in \hat{R}$ has a unique expression of the form

$$r = \sum_{m \in \mathbb{N}^n} \delta(r_m) t^m$$

for elements $r_m \in R/I$.

NOTE: There is no noetherian hypothesis here.
Existence for $r \in R$ since $\pi(r - z(r)) = 0$ we have

$$r - z(r) \in \mathbb{I}^\hat{R} \implies r - z(r) = \sum_{i=1}^{\hat{R}} a_i f_i \quad \text{some } a_i \in \hat{R}$$

But $a_i - z(ai) \in \mathbb{I}^\hat{R} \implies a_i - z(ai) = \sum_{j=1}^{\hat{R}} a_j f_j$

$$r = z(r) + \sum_{i=1}^{\hat{R}} \left\{ z(ai) + \sum_{j=1}^{\hat{R}} a_j f_j \right\} f_i$$

$$= z(r) + \sum_{i=1}^{\hat{R}} z(ai) f_i + \sum_{i,j} a_j f_i f_j$$

continuing in this way produces a series converging to $r$.

Uniqueness follows from quasi-regularity. Suppose to the contrary that

$$\sum_{M} z(r_m) t^M = 0 \quad \text{in} \quad \hat{R}$$

with not all $r_m$ zero in $R/I$. Let $m := \min\{ |M| \mid r_m \neq 0 \text{ in } R/I \}$. Since the Cauchy sequence $\{ \sum_{|M| \leq d} z(r_m) t^M \}_{d \in \mathbb{N}}$ converges to $0$, we can find $D$ s.t. for all $d > D$, 

$$\sum_{|M| \leq d} z(r_m) t^M \in (t, \ldots, t_n)^{m+1} \quad \text{in } R.$$ 

$$\sum_{|M|=m} z(r_m) t^M + \sum_{m < |M| \leq d} z(r_m) t^M$$

This implies $\sum_{|M|=m} z(r_m) t^M \in \mathbb{I}^{m+1}$, so $\sum_{|M|=m} r_m t^M = 0$ in $\mathbb{I}^m/\mathbb{I}^{m+1}$ and by defn of quasi-regularity this forces $r_m \in I$ so $r_m = 0 \text{ in } R/I$ for all $M$ with $|M| = m$. But this is a contradiction, so we arrive at the desired uniqueness of the representation ($\dag$).
This shows there is an isomorphism of $k$-modules

$$\hat{R} \xrightarrow{\cong} \prod_{n \in \mathbb{N}} \frac{R}{I}$$

$$r \mapsto \left( r_{m} \right)_{m}$$

But then since $\frac{R}{I}$ is f.p. over $k$, $\mathbb{Z}^{k}$ is the isomorphism

$$\frac{R}{I} \otimes_{k} k[t] \cong \frac{R/\mathfrak{I}}{\otimes_{k}} \prod_{m} k \cong \prod_{m} \left( \frac{R}{I} \otimes_{k} k \right) \cong \prod_{m} \frac{R}{I} \cong \hat{R} \quad \Box$$

**Corollary** If $\frac{R}{I}$ is a f.g. projective $k$-module there is a $k$-linear connection

$$\nabla : \hat{R} \longrightarrow \hat{R} \otimes_{k[t]} \prod_{i}^{m} k[t] \otimes_{k} \frac{R}{I}$$

$$\nabla (r f(t)) = \nabla(r) f(t) + r \otimes df.$$  

Relative to a fixed connection we introduce $\frac{\partial}{\partial t_{i}} \in \text{End}_{k}(\hat{R})$ via $\nabla = \sum_{j} \frac{\partial}{\partial t_{j}} \cdot dt_{j}$.

**Proof** $\hat{R}$ is a direct summand of a finite direct sum of copies of $k[t]$, which has a connection. Choosing a section $\mathfrak{B}$ we obtain a connection $\nabla_{\mathfrak{B}}$,

$$\nabla_{\mathfrak{B}}(r) = \sum_{m \in \mathbb{N}} \sum_{j=1}^{m} M_{j} \mathfrak{B} \left( r_{m} \right) t^{m-j} \otimes dt_{j}. \quad \Box$$
The "formal" tubular neighborhood is not an actual tubular neighborhood, so the intuition can be misleading, but one can think of the connection $\nabla$ as differentiation in the directions normal to $Y \subset X$, i.e.

The plan for the remainder of the lecture is to use this connection to define residues and provide a natural homotopy equivalence between the Koszul complex of $a = (a_1, \ldots, a_n)$ and its cohomology $R/I$. But first, an example:

**Example** Let a field, $R = k[x]$, $a = x^d$. Choose the $k$-linear section

$$\beta : R/I \to R, \quad \beta(x^i) = x^i \quad 0 \leq i \leq d-1,$$

and let $\nabla$ be the associated connection, with $\frac{\partial}{\partial t_i} : R \to R$. Then ($d > 2$)

$$\frac{\partial}{\partial t_i}(x^2 + x^d + 1) = \frac{\partial}{\partial t_i}(x^2 + [x^2] \cdot 1 + [x] \cdot x^d) = x.$$

**Remark** Since the $t_i$ are the coordinates in $(R/I)^*\!$, arising from the $a_i$, we will often write $\frac{\partial}{\partial a_i}$ for $\frac{\partial}{\partial t_i}$ where it will not cause confusion.

**Proposition** Let $a = (a_1, \ldots, a_n)$ be a sequence in $R$. Then

(i) $a$ regular $\Rightarrow$ $a$ quasi-regular

(ii) If $R$ is noetherian, $a$ is quasi-regular $\iff$ the Koszul complex on $a$ is exact except in degree zero.

(iii) If $(R, m, k)$ is local and $a \subseteq m$ then regular $\iff$ quasi-regular.
Suppose $a \in R$ quasi-regular and $R/I$ f.g. projective over $k$. Given $r_0, r_1, \ldots, r_n \in R$ we define

$$\text{Res}_{R/k} \left( \frac{r_0 \, dr_1 \cdots dr_n}{a_1 \cdots a_n} \right) := tr^{k}_{R/I} \left( r_0 \left[ \nabla, r_1 \right] \cdots \left[ \nabla, r_n \right] \right) \in k$$

\[ \hat{R} \longrightarrow \hat{R} \otimes \mathcal{O}^1 \longrightarrow \hat{R} \otimes \mathcal{O}^2 \longrightarrow \cdots \longrightarrow \hat{R} \otimes \mathcal{O}^n \cong \hat{R} \]

Remark: This is essentially the definition of residues given by Lipman, rephrased to use connections. He proves the basic properties (e.g., the transformation rule) for these residues.

Example: $\text{Res}_{k[x]/k} \left( \frac{x^i \, dx}{x^d} \right) = \text{tr}_{k[x]/k} \left( x^i \left[ \frac{\partial}{\partial x} x - x \frac{\partial}{\partial x} \right] \right) = \text{tr}_{k[x]/k} \left( x^i \frac{\partial^2}{\partial x^2} x \right)$

\[ \begin{pmatrix} 1 & x & x^2 & \ldots & x^{d-1} \\ 0 & 1 & & & \\ & \ddots & \ddots & \ddots \\ & & 0 & 1 & \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix} = \delta_{i,d-1}. \]

Remark: If $r_0 \in I$ the residue is zero.
Remark Suppose \( r_i = z(s_i) \) for \( 1 \leq i \leq n \) (this is typical). Then the residue is

\[
\sum_{i_1, \ldots, i_n} \text{tr}_{R/I}^k \left( z(s_0) \cdot \frac{\partial}{\partial x_{i_1}} \cdot z(s_1) \cdot \ldots \cdot \frac{\partial}{\partial x_{i_n}} \cdot z(s_n) \right) \quad (*)
\]

If \( \tilde{z}^*: R/I[[x_0]] \to \hat{R} \) were an isomorphism of rings, we would have (in \( \hat{R} \))

\[ z(s) \tilde{z}(s') = z(ss') \] and so the residue would have to be zero. The residue reflects the failure of \( z^* \) to be a ring isomorphism, or more precisely, it is an expression in the \( t \)-derivatives of the following tensor:

Define with the same hypotheses as in the definition of residues, let \( T \) be the tensor

\[ T \in (R/I)^* \otimes (R/I)^* \otimes (R/I)^* \otimes k[[t]] \]

given by the \( k \)-linear map

\[
\begin{array}{ccc}
R/I \otimes_k R/I & \xrightarrow{z \otimes z} & \hat{R} \otimes_k \hat{R} & \xrightarrow{\text{mult}} & \hat{R} & \xrightarrow{\bar{z}} & R/I \otimes_k k[[t]].
\end{array}
\]

Example \( R = k[z] \), \( a = xa \), \( R/I = \bigoplus_{i=0}^{d-1} kx^i \), for \( 0 \leq i, j \leq d-1 \) write

\[
i + j = qd + r \quad 0 \leq r, q < d
\]

\[
z(x^i)z(x^j) = x^{i+j} = x^r(x^d)^q = z(x^r)t^q
\]

Hence

\[
T(x^i, x^j) = x^r \otimes t^q, \quad T_{k, l}^{ij} = \delta_{k, x(r, i)} \delta_{l, q(1, j)}
\]
Constructions on the Koszul complex  Let \( a \) be a quasi-regular sequence, and assume \( R/I \) is f.g. projective over \( k \). Consider the quasi-isomorphism
\[
\mathcal{K} := (\wedge (k \Theta_1 \cdots k \Theta_n) \otimes_R \hat{R}, \ d_k = \sum_{i=1}^n a_i \Theta_i^* ) \quad \overset{\pi}{\rightarrow} \quad R/I = (R/I, 0).
\]

In fact this is a homotopy equivalence over \( k \). Choose \( \beta \) a \( k \)-linear retraction and \( \nabla \) the associated homotopy. We view \( \nabla = \sum_i \frac{\partial}{\partial t_i} \Theta_i \) as an odd \( k \)-linear operator on \( \mathcal{K} \). Then
\[
[\nabla, d_k] (r \Theta_{i_1} \cdots \Theta_{i_p}) = (\nabla d_k + d_k \nabla)(r \Theta_{i_1} \cdots \Theta_{i_p}) = \nabla (\sum_{j=1}^p (-1)^j r a_{i_j} \Theta_{i_1} \cdots \Theta_{i_j} \cdots \Theta_{i_p}) + d_k (\sum_{k=1}^n \frac{\partial}{\partial t_k} (r) \Theta_k \Theta_{i_1} \cdots \Theta_{i_p})
\]

by def' of a connection
\[
\frac{\partial}{\partial t_k} (r a_{i_j}) = a_{i_j} \frac{\partial}{\partial t_k} (r)
\]

unless \( k = l_j \)
\[
+ \sum_{j,k=1}^n \frac{\partial}{\partial t_k} (r) a_k \Theta_{i_1} \cdots \Theta_{i_j} \cdots \Theta_{i_p}
\]

\[
= \sum_j \left\{ \frac{\partial}{\partial t_j} (r a_{i_j}) - \frac{\partial}{\partial t_{i_j}} (r) a_{i_j} \right\} \Theta_{i_1} \cdots \Theta_{i_p}
\]

+ \sum_{k=1}^n \frac{\partial}{\partial t_k} (r) a_k \Theta_{i_1} \cdots \Theta_{i_p}
\]

\[
= \left\{ \sum_j r + \sum_k \frac{\partial}{\partial t_k} (r) a_k \right\} \Theta_{i_1} \cdots \Theta_{i_p} = \left\{ p + d_k \nabla \right\} (r) \Theta_{i_1} \cdots \Theta_{i_p}.
\]
Observe that
\[
dk \nabla \left( \sum_{m} \partial (m) t^m \right) = \sum_{m} \sum_{j} M_{j} \partial (m) t^{m-q_j}
\]
\[
= \sum_{m} \sum_{j} M_{j} \partial (m) t^{m}
\]
\[
= \sum_{m} \text{Im} \partial (m) t^{m}
\]

**Lemma** \( \text{Im} (dk \nabla^o : R \to R) \) is the ideal \( I \).

**Lemma** Let \( K_{\geq 1} \subseteq K \) denote the submodule spanned by \( r \otimes \partial_1 \cdots \partial_p \) with \( p \geq 1 \). Then \( K_{\geq 1} \) is closed under \([\nabla, dk] \) and the \( k \)-linear map \([\nabla, dk] : K_{\geq 1} \to K_{\geq 1} \) is invertible.

**Proof** \([\nabla, dk]^{-1} (r \partial_1 \cdots \partial_p) = \sum_{m} \frac{1}{p + \text{Im}(m)} \partial(m) t^{m} \partial_1 \cdots \partial_p \)

**Theorem** Consider the diagram of \( k \)-linear maps
\[
H \subset (K, dk) \xrightarrow{\pi} (R/I, O)
\]
where \( H = [\nabla, dk]^{-1} \circ \nabla \). Then we have

(a) \( \pi, \partial \) are cochain maps (of \( k \)-complexes)

(b) \( \pi \partial = 1 \)

(c) \( \partial \pi = 1_k - [dk, H] \)

(d) \( H^r = 0, H^2 = 0, \pi H = 0 \).

This kind of situation is called a strong deformation retract.
Proof. We prove (c). Observe that, writing $J = [\nabla, dk]$

\[
dk J = dk[\nabla, dk] = dk(\nabla dk + dk \nabla)
= dk \nabla dk = [\nabla, dk]dk = Jdk
\]

Write $K_p \subseteq K$ for the submodule spanned by $r \otimes \Omega_i \cdots \Omega_p$, $r \in \hat{R}$. Then on $K_p$ for $p > 0$ we have by (\#) that $dk = Jdk J^{-1}$ on $K_p$, and hence

\[
[dk, H] = dk H + H dk
= dk J^{-1} \nabla + J^{-1} \nabla dk
= J^{-1} dk J^{-1} \nabla + J^{-1} \nabla dk
= J^{-1} (dk \nabla + \nabla dk) = J^{-1}J = 1
\]

whereas on $K_0$ we have for $r \in \hat{R}$ that $r - 2(r) \in I$ and for $x \in I$ we know there exists $y \in \hat{R}$ with $x = dk \nabla (y)$. Then

\[
(1 - dk H)(x) = (1 - dk H)dk \nabla (y)
= dk \nabla (y) - dk J^{-1} \nabla dk \nabla (y)
= dk \nabla (y) - dk J^{-1} (J - dk \nabla) \nabla (y)
= 0.
\]

But this shows that on $K_0$,

\[
(1_k - [dk, H])(r) = (1_k - dk H)(r)
= (1_k - dk H)(\delta(r))
= \delta(r).
\]
References


  “the purpose of this paper is to provide an elementary development of the theory of residues.”