Let $k$ be a commutative ring, and $A = k[x_1, \ldots, x_n]$ a polynomial ring. In the category of $A$-$A$-bimodules there are two natural projective resolutions of $A$ as a bimodule (i.e. the diagonal)

\[
\begin{array}{ccc}
\text{Bar complex (noncommutative forms)} & B & \pi_B \\
\Psi & (\Phi) & A \\
K & \pi_K & \\
\end{array}
\]

Koszul complex (commutative forms)

By standard homological algebra there exist morphisms of complexes, $\Phi$ and $\Psi$ such that

\[
\pi_K \circ \Psi = \pi_B, \quad \pi_B \circ \Phi = \pi_A, \quad \Phi \circ \Psi \simeq 1_B, \quad \Psi \circ \Phi \simeq 1_K.
\]

The aim of this talk is, starting from only a knowledge of basic (homological) algebra, to first of all define $B, K$ and then describe explicitly $\Phi, \Psi$. The map $\Phi$ is standard (e.g. from the proof of Hochschild–Kostant–Rosenberg’s theorem) but the explicit description of $\Psi$ as a chain map seems less well-known. On the latter point we are following the papers


and the concept of

**Noncommutative differential forms** (over a commutative ring \( \mathbb{k} \))

A differential graded algebra (DGA) is a monoid in the monoidal category of \( \mathbb{Z} \)-graded complexes of \( \mathbb{k} \)-modules \((\text{Ch}_\mathbb{Z}(\mathbb{k}), \otimes, \mathbb{I}=\mathbb{k})\), that is, a DGA is a tuple \((A, \otimes, m, u)\) where \((A, \otimes) \in \text{Ch}_\mathbb{Z}(\mathbb{k})\) and

\[
m: A \otimes A \to A, \quad u: \mathbb{k} \to A \quad \left( A = \bigoplus_{i \in \mathbb{Z}} A^i \right)
\]

are morphisms of complexes satisfying associativity and unit constraints.

**Remarks**

1. \(1_A := u(1_k)\) is a closed element of \(A^0\). \((\partial^i: A^i \to A^{i+1})\)

2. \(A^0\) is a \(\mathbb{k}\)-algebra with \(m^0_{A^0} \otimes A^0\) and \(1_A\).

3. If \((A, m, u)\) is a \(\mathbb{k}\)-algebra then \((A, 0, m, u)\) is a DGA.

“**Lemma**” There is an adjoint pair of functors \((I = \text{inclusion})\)

\[
\text{DGA}(\mathbb{k}) \xleftarrow{(-)^0} \text{Alg}(\mathbb{k}) \xrightarrow{(-)^0} \mathbb{k}\text{-algebras}
\]

\[
(-1)^0: \text{DGA}(\mathbb{k}) \to \text{Alg}(\mathbb{k}) \quad \text{and} \quad \text{I}: \text{Alg}(\mathbb{k}) \to \text{DGA}(\mathbb{k})
\]

\(\text{DGA}(\mathbb{k})\) is a category of differential graded \(\mathbb{k}\)-algebras, and \(\text{Alg}(\mathbb{k})\) is a category of \(\mathbb{k}\)-algebras (associative, unital, possibly not commutative).
"Proof." The unit is the identity \( \eta_A : A \to I(A)^\circ = A \). This is natural, and given any algebra map \( \alpha : A \to B^\circ \) for a DGA \( B \),

\[
\tilde{\alpha} : I(A) \to B \\
\tilde{\alpha}_i = \begin{cases} \\
0 & i \neq 0 \\
\alpha & i = 0
\end{cases}
\]

is the unique morphism of DGA\(\text{s} \) making \( \tilde{\alpha} = \text{id} \) on \( A \) and \( (\tilde{\alpha})^\circ \) on \( I(A)^\circ \). However, this is not a correct proof (the lemma is false).

Commute. \( \square \)

The problem with this "proof" is of course that \( \tilde{\alpha} \) is not a morphism of DGA\(\text{s} \) as soon as \( \text{Im}(\alpha) \neq Z^\circ(B) \), since if \( \partial B \alpha(a) \neq 0 \) then

\[
\partial_B \tilde{\alpha}(a) \neq 0 = \tilde{\alpha} \partial I(A)(a)
\]

To fix this and find a right adjoint to \((-)^\circ \) we need "\( DA \)" fitting in a diagram

\[
\begin{array}{ccc}
0 & \rightarrow & 1 & \rightarrow & \cdots \\
\downarrow & & & & \downarrow \\
A & \rightarrow & dA & \rightarrow & \cdots \\
\downarrow & & & & \downarrow \\
B^\circ & \rightarrow & B^1 & \rightarrow & \cdots \\
\end{array}
\]

\( \partial_B \alpha(a) \neq 0 \rightarrow \tilde{\alpha} \partial I(A)(a) \)
Define The DGA of noncommutative differential forms over an algebra $A$ is, if it exists, a pair $(\Omega A, \gamma)$ consisting of:

- a DGA $\Omega A$,
- a morphism of algebras $\gamma : A \to (\Omega A)^{\circ}$

which is universal, in the sense that if $B$ is a DGA and $\alpha : A \to B^{\circ}$ an algebra morphism, there is a unique morphism of DGAs $\tilde{\alpha} : \Omega A \to B$ such that the following diagram commutes:

$$
\begin{array}{ccc}
A & \xrightarrow{\gamma} & (\Omega A)^{\circ} \\
\downarrow{\alpha} & & \downarrow{\gamma^{\circ}} \\
B^{\circ} & & (\tilde{\alpha})^{\circ}
\end{array}
$$

equivalently, unique s.t. $\tilde{\alpha} \circ \gamma = \alpha$, viewing $d$ as a map into $\Omega A$.

Theorem (Cuntz–Quillen) $(\Omega A, \gamma)$ exists for all $A$, so there is an adjunction

$$
\begin{array}{ccc}
\text{DGA}(k) & \xleftarrow{(-)^{\circ}} & \text{Alg}(k) \\
\Omega & \xrightarrow{(-)^{\circ}} & \text{Alg}(k)
\end{array}
$$

That is, we have natural bijections

$$
\text{Hom}_{\text{Alg}(k)}(A, B^{\circ}) \cong \text{Hom}_{\text{DGA}(k)}(\Omega A, B).
$$

Remark. The morphism of algebras $A \xrightarrow{1_A} I(A)^{\circ} = A$ induces a morphism of DGAs

$$
\omega : \Omega A \to I(A) \quad \text{s.t.} \quad \omega \circ \gamma = 1_A,
$$

i.e. $\gamma : A \to (\Omega A)^{\circ}$ has a section $\omega : (\Omega A)^{\circ} \to A$, in $\text{Alg}(k)$.
Proof First observe that in any DGA \((\mathcal{A}, d)\) with \(A \subseteq \mathcal{A}^0\) as a subalgebra

1. \(d(a_0a_1 \cdots a_n) = da_0a_1 \cdots a_n\)

2. \((a_0a_1 \cdots a_n)(a_{n+1}a_{n+2} \cdots a_k)\)

\[\begin{align*}
&= (-1)^n a_0a_1a_2 \cdots a_k \\
&\quad + \sum_{i=1}^{n} (-1)^{n-i} a_0a_1 \cdots d(a_i a_{i+1}) \cdots a_k
\end{align*}\]

The second formula results from multiple applications of

\[d(b)c = d(bc) - bd(c).\]

Thus \(\mathcal{A}\) has a sub-DGA-algebra spanned by expressions \(a_0a_1 \cdots a_n\). Since \(\lambda \in k\) implies \(d(\lambda \cdot a) = 0\) (since \(1_a = 1_\mathcal{A}\)) this leads us to define a \(\mathbb{Z}\)-graded \(k\)-module \((\otimes = \otimes_k)\)

\[\mathcal{A} := \bigoplus_{n \geq 0} A \otimes \bar{A}^n\]

\[\bar{A} = A/k\]

and make \((\mathcal{A}, d)\) a complex with

\[d(a_0 \otimes \bar{a}_1 \otimes \cdots \otimes \bar{a}_n) = 1 \otimes a_0 \otimes \bar{a}_1 \otimes \cdots \otimes \bar{a}_n.\]

Now there is a clever trick to induce the desired DGA structure. Define

\[E = \text{Hom}^*_k((\mathcal{A}, d), (\mathcal{A}, d))\]

\((\text{DGA-algebra})\) with differential \([d, -]\)

\[\begin{align*}
[d, f] &= d \circ f - (-1)^{|f|} f \circ d \\
(f : \mathcal{A} \to \mathcal{A} \text{ is a degree} \ n \text{ map of} \ \mathbb{Z}\text{-graded modules})
\end{align*}\]
Define \( k \)-linear maps
\[
\ell : A \to E^0, \quad \ell(a)(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = a_0 \otimes a_i \otimes \cdots \otimes a_n
\]
\[
\ell_* : \Lambda A \to E, \quad \ell_*(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = \ell(a_0)[d, \ell(a_1)] \cdots [d, \ell(a_n)]
\]

Note that
\[
[d, \ell(a)](1 \otimes a_1 \otimes \cdots \otimes a_n) = d(a_0 \otimes a_1 \otimes \cdots \otimes a_n)
\]
\[
= 1 \otimes a_i \otimes a_{i+1} \otimes \cdots \otimes a_n.
\]

By (1), (2) we have \( \text{Im}(\ell_*) \subseteq E \) is the DG-subalgebra spanned by \( \ell(A) \).

Now define \( \mathcal{T} : E \to \Lambda A \) by \( \mathcal{T}(f) = f(1) \), this is \( k \)-linear and degree zero. Moreover, using (\( \ast \)),
\[
\mathcal{T} \ell_*(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = \{ \ell(a_0)[d, \ell(a_1)] \cdots [d, \ell(a_n)] \} \quad (1)
\]
\[
= \ell(a_0)[d, \ell(a_1)] \cdots [d, \ell(a_n)](1 \otimes a_n)
\]
\[
= \ell(a_0)(1 \otimes a_i \otimes \cdots \otimes a_n)
\]
\[
= a_0 \otimes a_i \otimes \cdots \otimes a_n.
\]
Hence \( \mathcal{T} \ell_* = 1_{\Lambda A} \) and \( \ell_* \) is injective. Since \( \ell_*(\Lambda A) \subseteq E \) was a sub-DGA this equips \( \Lambda A \) with a unique DG-structure s.t. \( \ell_* \) is a map of DGAs.

So far we have defined a DGA \( (\Lambda A = \bigoplus_{n \geq 0} A \otimes A^\otimes_n, d) \), and in this DGA
\[
a_0da_1 \cdots da_n = a_0 \otimes a_i \otimes \cdots \otimes a_n. \quad \text{(apply } \ell_* \text{ to both sides)}
\]
\[
\therefore \text{the structure of this DGA is completely determined by (1), (2).}
It remains to show the algebra map

$$\gamma = 1_A : A \longrightarrow (\wedge A)^\circ = A$$

is universal: suppose $B$ is a DGA and $u : A \rightarrow B^\circ$ an algebra map. Then

$$\tilde{u} : \wedge A \longrightarrow B, \quad a_0 \otimes \bar{a}_1 \otimes \cdots \otimes \bar{a}_n \mapsto u_{a_0} \partial_B(u_{a_1}) \cdots \partial_B(u_{a_n})$$

is a morphism of DGAs using (1), (2) and is clearly unique s.t. $\tilde{u}|_A = u$. $\square$

Example: $A = k[x]$, $\wedge A = \left( k[x] \longrightarrow k[x] \otimes k[x] \longrightarrow k[x] \otimes (x k[x]) \otimes \cdots \right.$

$$x^i \longmapsto 1 \otimes x^i$$

$$x^i \otimes x^j \longmapsto 1 \otimes x^i \otimes x^j, \quad j \geq 1$$

$$\vdash \quad H^0(\wedge A) \cong k, \quad H^n(\wedge A) = 0 \quad n > 0.$$

Remark: If $A$ is commutative, $(\wedge A)^\text{sup} \cong \left( \wedge A \otimes A/k, \quad d \right)$ as DGAs. $^{\text{quotient making it graded commutative and } \partial^2 = 0}$

Kähler differentials
The Bar complex $\mathbb{B}$ (normalised) is the complex

$$\mathbb{B} = \left( \bigwedge A \bigotimes A, b' \right)$$

$$b'(wda \otimes a') = (-1)^{|w|}(wa \otimes a' - \omega \otimes aa') \quad a, a' \in A$$

More explicitly, $\mathbb{B}_n = \bigwedge^n A \bigotimes A = A \bigotimes \bar{A}^\otimes \otimes A$ and using (2),

$$b'(a_0 \otimes \bar{a}_1 \otimes \cdots \otimes \bar{a}_n \otimes a_{n+1}) = \sum_{i=0}^{n} (-1)^i a_0 \otimes \bar{a}_1 \otimes \cdots \otimes \bar{a}_i a_{i+1} \otimes \cdots \otimes a_{n+1}$$

Lemma The complex of $\mathcal{A}$-A-bimodules

$$\cdots \to A \bigotimes \bar{A}^\otimes A \xrightarrow{b'} A \bigotimes \bar{A} \bigotimes A \xrightarrow{b'} A \bigotimes A \xrightarrow{b' = m} A \to 0$$

is exact, in fact contractible as a complex of right $A$-modules.

Proof $(\mathcal{A}, d)$ is a DGA, and $d \otimes 1$ is a right $A$-linear degree $-1$ operator on $\mathbb{B} = \mathcal{A} \bigotimes A$. One checks directly that

$$b'd + db' = 1 \quad \text{on } \mathbb{B}_n, \quad n \geq 0$$

where we include $b' : A \bigotimes A \to A$ for $n = 0$. \qed

Question How to see $b'$ as an operator on $\bigwedge A \bigotimes A = \bigwedge A (A \bigotimes A)$ purely from the universal property? I don't know.
The Koszul complex $IK$

Now we specialise to $A = k[x_1, \ldots, x_n]$ so $B \to A$ is a free resolution.

Another free resolution is the Koszul complex $(A^e = A \otimes A, |\cdot| = 1)$

$$IK = \left( \bigwedge (k \Theta_1 \otimes \cdots \otimes k \Theta_n) \otimes A^e, \sum_{i=1}^n \Theta_i^* \otimes [x_i \otimes 1 \otimes x_i] \right)$$

$$\Theta_i^*(\alpha \beta) = \Theta_i^*(\alpha) \beta + (-1)^{|\alpha|} \alpha \Theta_i^*(\beta)$$

$$\Theta_i^*(\Theta_j) = \delta_{ij}$$

The first few terms look like

$$0 \to A \Theta_1 - \Theta_n \to \cdots \to \bigoplus_{i=1}^n \Theta_i^* A \to A^e \to A \to 0$$

Remark. With exterior multiplication, $IK$ is a DGA, $m: IK \to A$ is a morphism of DGAs.

We have now two free resolutions of $A$ as an $A$-$A$-bimodule.

$$B \xrightarrow{\pi_B} A, \quad IK \xrightarrow{\pi_K} A$$

It is sometimes useful (in e.g. perturbation theory, which is the context of [CM] ) to know formulas for chain maps $\Phi: IK \to B$, $\Psi: B \to IK$ lifting the identity on $A$. One reason to care about $\Psi$ is that lifting into $B$ as in the diagram below is often easier than lifting into $IK$, as noncommutative differential forms are better behaved than ordinary differential forms. One can obtain an explicit lifting to $IK$ by first lifting to $B$ (i.e. $F$) and then composing to get $\Psi \circ F$.

But only if one knows a formula for $\Psi$!
Now for the formulas

The induced homotopy equivalence \( \Psi : LB \rightarrow IB \) is

\[
\Psi \left( (\circ_i \circ_{i-1} \cdots \circ_2) \cdot r \right) = \sum_{i \in S_p} (-1)^i d x_{i(i-1) \cdots i_2} \cdot d x_{i_2} \cdot \cdots \cdot d x_{i_p} \cdot \circ_{i} \circ_{i-1} \cdots \circ_2 \cdot r
\]

and its homotopy inverse is \( \Psi : LB \rightarrow IK \),

\[
\Psi \left( r d x_{i} \cdots d x_{i_p} \circ r' \right) = \sum_{1 \leq i_1 < \cdots < i_p \leq n} \left( r \circ r' \right) \prod_{k=1}^{p} \partial_{[i_k]} (f_{i_k}) \cdot \circ_{i_1} \cdots \circ_{i_p}
\]

where we use the divided difference operators

\[
\partial_{[i]} : A = A[x] \rightarrow A[x, x'] = A^e
\]

\[
\partial_{[i]} (f) = f(x_1, \ldots, x_{i-1}, x_i, \ldots, x_n) - f(x_1, \ldots, x_{i-1}, x_i', \ldots, x_n) \quad x_i - x_i'
\]

The formula for \( \Psi \) is standard, \( \Psi \) is from [SW], [CM]. To prove \( \Psi \) are mutually inverse up to homotopy one just checks by hand the formulas give cochain maps, and that they induce \( 1_A \).

**Lemma** \( \Phi, \Psi \) are morphisms of DGAs between \( IK \) and \( (B, b', x) \) where \( x \) is the shuffle product.
Given an $A$-$A$-bimodule $M$, the Hochschild homology is defined as

$$\text{Hochschild homology } \quad \text{HH}_n(A, M) := \text{H}_n(M \otimes_A \text{IB})$$

and the Hochschild cohomology is defined as

$$\text{Hochschild cohomology } \quad \text{HH}^n(A, M) := \text{H}^n(\text{Hom}_A(\text{IB}, M)).$$