Singular Chern Classes

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This note contains references for the last part of my talk on the local Euler obstruction. Recall the notation: all schemes are finite type over $\mathbb{C}$ and for a variety V the Chern-Mather class, a cycle class in $A_*(X)$, is written $c^M(V)$ (this is the notation of [Beh05], Kennedy [Ken90] writes $\hat{c}(V)$). The main theorem of the talk was the equality, for any proper scheme X and subvariety $V \subset X$, of the two invariants

$$\chi(X, \mathrm{Eu}_V) = \int_X c^M(V). \tag{0.1}$$

This is (the restriction to schemes of) Proposition 1.12 in [Beh05] and is a special case of MacPherson's main theorem in [Mac74], which states that the Schwartz-MacPherson Chern class, interpreted as a morphism of abelian groups $c^{SM} : \mathrm{Con}(X) \longrightarrow A_*(X)$ for schemes X of finite type over $\mathbb{C}$, is natural with respect to proper pushforward. Recall that c^{SM} is defined so as to make the diagram

$$\begin{tikzcd} Z_*(X) \arrow[rr, "\mathrm{Eu}", "\cong"'] \arrow[dr, "c^M"'] & & \mathrm{Con}(X) \arrow[dl, "c^{SM}"] \\ & A_*(X) & \end{tikzcd}$$

commute, where Eu is the local Euler obstruction. If X is proper and $f : X \longrightarrow \mathrm{Spec}(\mathbb{C})$ is the structure morphism then evaluating naturality of c^{SM} with respect to f on the constructible function Eu_V produces (0.1). MacPherson's original proof of naturality in [Mac74] uses a topological definition of the local Euler obstruction. An algebraic definition of the obstruction in terms of the Nash blowup and Segre classes was given by Gonzalez-Sprinberg and Verdier [Gon81]. This is the definition adopted in our main reference [Ken90] and presented in the seminar.

Using this algebraic definition of the Euler obstruction we proved naturality of c^{SM} in the special case of a morphism $f : X \longrightarrow \mathrm{Spec}(\mathbb{C})$, following [Ken90]. In outline: one introduces a third functor on the category of finite type schemes and proper maps associating to X the abelian group $\mathcal{L}X$ of conical Lagrangian cycles over X. It is easy to see that there is an isomorphism

$$\mathrm{Con}(X) \cong \mathcal{L}X \tag{0.2}$$

and one defines a pushforward map of Lagrangian cycles; see [Sab85, Ken90]. Kennedy then defines a map $\varphi : \mathcal{L}X \longrightarrow A_*(X)$ which he proves to be natural. However to deduce (0.1) we need to know that the isomorphism (0.2) is also natural, in order to deduce that the Lagrangian pushforward of the cycle originating from a subvariety $V \subset X$ to a point recovers the topological Euler characteristic $\chi(X, \mathrm{Eu}_V)$. For this Kennedy refers (in the introduction of [Ken90]) to Sabbah [Sab85] and the key point in Sabbah's proof is an index theorem of Dubson.

This index theorem goes back to an old preprint [Dub81], the announcement of which [Dub84] contains the statement of the needed result [Dub84, Theorem 2, p.115] but no proofs. The paper of Schürmann [Sch04] also gives the statement (Corollary 0.1) together with a useful account of subsequent generalisations and alternative proofs, in for example Kashiwara and Schapira [Kas90, Corollary 9.5.2, p.384] and [Gin86] (the introduction to which I highly recommend!).

References

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