Polynomial functors and logic

In this note we tentatively establish a connection between <u>strict polynomial functors</u> $F : \mathrm{mod}_k \longrightarrow \mathrm{mod}_k$, for k a char. 0 alg. closed field, and <u>linear logic</u>. Let $\mathrm{rep}\,\Gamma^d_k$ denote the category of finitely generated polynomial functors of degree d, and $!\mathrm{mod}_k$ the "universal enrichment" of mod_k over the category of k-coalgebras (see below). We show that there is a fully faithful functor

$$\mathrm{rep}\,\Gamma^d_k \hookrightarrow [\,!\mathrm{mod}_k, \mathrm{mod}_k]_k \qquad (1.1)$$

where for k-linear categories $\mathcal{C}, \mathcal{C}'$ we write $[\mathcal{C},\mathcal{C}']_k$ for the category of k-linear functors. The connection to <u>linear logic</u> is as follows. Consider the 2-category $\mathcal{V}_k$ of k-linear categories, k-linear functors, and natural transformations. This is (almost) closed monoidal using $[-,-]_k$ and $\otimes_k$ (but modulo set-theoretic issues). For any k-linear category $\mathcal{C}$ we denote by $!\mathcal{C}$ the universal k-linear category enriched over Coalg_k which maps to $\mathcal{C}$ in $\mathcal{V}_k$, via a functor

$$!\mathcal{C} \xrightarrow{d_{\mathcal{C}}} \mathcal{C}. \qquad (1.2)$$

The structured tuple $(\mathcal{V}_k, \otimes_k, [-,-]_k, !)$ should give a semantics of intuitionistic first order linear logic (see [4] for references), although the theory of 2-categorical semantics is still undeveloped. More precisely, I would expect the <u>category</u> $\mathcal{B}(\mathcal{V}_k)$ (with the same objects as $\mathcal{V}_k$ and arrows 2-isomorphism classes of 1-morphisms) to carry a semantics of linear logic as defined in [5]. The point being that linear logic <u>expresses all formal constructions</u> of functors possible using $\otimes_k$, $[-,-]_k$ and $!$.

(x an atomic formula)

<u>For example</u> any term (i.e. algorithm) π in linear logic of type $!(!x \multimap !x) \multimap (!x \multimap !x)$ gives rise, after interpreting $[\![x]\!] := \mathrm{mod}_k$, to a k-linear functor ($\multimap$ is the logician's Hom)

$$[\![\pi]\!] : ![\,!\mathrm{mod}_k, !\mathrm{mod}_k]_k \longrightarrow [\,!\mathrm{mod}_k, !\mathrm{mod}_k]_k \qquad (1.3)$$

If $F \in \mathrm{rep}T^d_k$ then under (1.1) there is an associated k-linear $!\mathrm{mod}_k \longrightarrow \mathrm{mod}_k$ which by the universal property lifts to a k-linear functor $!\mathrm{mod}_k \longrightarrow !\mathrm{mod}_k$ we denote by $\widetilde{F}$. Then, since $\mathrm{ob}(!\mathscr{C}) = \mathrm{ob}(\mathscr{C})$, $\widetilde{F}$ is also an object of $![!\mathrm{mod}_k, !\mathrm{mod}_k]_k$ and thus a valid input to the functor (1.3). The output might be, for example "F^2" meaning $\widetilde{F} \circ \widetilde{F}$.

Upshot Polynomial functors give a natural class of inputs to the 2-categorical semantics of linear logic in k-linear categories. This semantics gives an elaboration of many ways to construct new functors out of these basic examples, for example by iterating F (i.e. iterating $\widetilde{F}$). Maybe some of these constructions are interesting?

Following [1,2] we denote by $!V \xrightarrow{d_V} V$ the universal morphism in Mod_k to a vector space V (possibly infinite-dimensional) out of a cocommutative coassociative counital coalgebra (henceforth just coalgebra). We denote the counit and comultiplication by (we drop subscripts on d, c, Δ where it will not cause confusion)

$$c_V : !V \longrightarrow k, \quad \Delta_V : !V \longrightarrow !V \otimes !V. \qquad (\otimes = \otimes_k)$$

Let $\mathscr{C}$ be a k-linear category, by Sweedler and [1,2] we have (for $\mathscr{C}(a,b)$ finite-dimensional)

$$!\mathscr{C}(a,b) = \bigoplus_{f: a \to b} \mathrm{Sym}_f(\mathscr{C}(a,b)). \qquad \mathrm{Sym}_f = \mathrm{Sym} \qquad (1.5.1)$$

Our notation is, for $g_1, \ldots, g_k \in \mathscr{C}(a,b)$ to write

$$|g_1, \ldots, g_k\rangle_f := \overline{g_1 \otimes \cdots \otimes g_k} \in \mathrm{Sym}_f(\mathscr{C}(a,b)) \subseteq !\mathscr{C}(a,b). \qquad (1.5.2)$$

Remark Everything we say works for $\mathscr{C}(a,b)$ infinite-dimensional, with some care. Probably also outside char. 0 and algebraically closed. But general base rings are much less clear.

Notation mod_k – f.d. vector spaces $/k$
Mod_k – all vector spaces $/k$
Coalg_k – all cocommutative coassociative counital coalgebras $/k$.
$[-,-]_k$ – k-linear functors

1 Categories enriched over coalgebras

Our reference for categories enriched over a monoidal category is [3].

Given a category $\mathscr{C}$ enriched over Mod_k there is a natural way to produce a category $!\mathscr{C}$ enriched over the category Coalg_k of (counital, cocommutative) coalgebras, see for example [7].

Def^n Let $\mathscr{C}$ be a k-linear category. We define a new k-linear category $!\mathscr{C}$ by

$$\mathrm{obj}(!\mathscr{C}) = \mathrm{obj}(\mathscr{C}), \qquad (!\mathscr{C})(a,b) := !\,\mathscr{C}(a,b).$$

The composition $m_{!\mathscr{C}}^{a,b,c} : !\mathscr{C}(b,c) \otimes !\mathscr{C}(a,b) \longrightarrow !\mathscr{C}(a,c)$ is the unique morphism of coalgebras making the diagram below commute

$$\begin{tikzcd}
!\mathscr{C}(b,c) \otimes !\mathscr{C}(a,b) \arrow[r, "d \otimes d"] \arrow[d, dashed, "m_{!\mathscr{C}}^{a,b,c}"'] & \mathscr{C}(b,c) \otimes \mathscr{C}(a,b) \arrow[d, "\circ"] \\
!\mathscr{C}(a,c) \arrow[r, "d"'] & \mathscr{C}(a,c)
\end{tikzcd} \tag{2.1}$$

Note $!\mathscr{C}$ is enriched over Mod_k but not additive, as e.g. $\mathrm{End}_{!\mathscr{C}}(0) = !0 \cong k$.

Remark Using [1] it is easy to see that, given $f: a \to b$, $g: b \to c$ in $\mathscr{C}$ and corresponding $|\emptyset\rangle_f \in (!\mathscr{C})(a,b)$, $|\emptyset\rangle_g \in (!\mathscr{C})(b,c)$ that $|\emptyset\rangle_g \circ |\emptyset\rangle_f = |\emptyset\rangle_{gf}$.

Lemma $!\mathcal{C}$ is a k-linear category

Proof Associativity follows from commutativity of

$$\begin{tikzcd}
(\mathcal{C}^!(c,d)\otimes\mathcal{C}^!(b,c))\otimes\mathcal{C}^!(a,b) \arrow[ddr] \arrow[ddddd] & & & & & \mathcal{C}^!(c,d)\otimes(\mathcal{C}^!(b,c)\otimes\mathcal{C}^!(a,b)) \arrow[lllll] \arrow[ddl] \arrow[ddddd] \\
& & & \mathcal{C}(c,d)\otimes(\mathcal{C}(b,c)\otimes\mathcal{C}(a,b)) \arrow[ddl] \arrow[dddd, bend left] & & \\
& (\mathcal{C}(c,d)\otimes\mathcal{C}(b,c))\otimes\mathcal{C}^!(a,b) \arrow[dr] \arrow[dd] & & & \mathcal{C}^!(c,d)\otimes(\mathcal{C}(b,c)\otimes\mathcal{C}(a,b)) \arrow[ul] \arrow[dd] & \\
& & (\mathcal{C}(c,d)\otimes\mathcal{C}(b,c))\otimes\mathcal{C}(a,b) \arrow[dd] & & & \\
& \mathcal{C}(b,d)\otimes\mathcal{C}^!(a,b) \arrow[dr] & & & \mathcal{C}^!(c,d)\otimes\mathcal{C}(a,c) \arrow[dl] & \\
\mathcal{C}^!(b,d)\otimes\mathcal{C}^!(a,b) \arrow[ur] \arrow[rr] \arrow[ddrr] & & \mathcal{C}(b,d)\otimes\mathcal{C}(a,b) \arrow[d] & \mathcal{C}(c,d)\otimes\mathcal{C}(a,c) \arrow[dl] & & \mathcal{C}^!(c,d)\otimes\mathcal{C}^!(a,c) \arrow[ll] \arrow[ul] \arrow[ddlll] \\
& & \mathcal{C}(a,d) & & & \\
& & \mathcal{C}^!(a,d) \arrow[u] & & &
\end{tikzcd}$$

Given $a\in ob(\mathcal{C})$ we define $1_a^{!\mathcal{C}} \in\, !\mathcal{C}(a,a)$ to be $1_a^{!\mathcal{C}} := |\phi\rangle_{1_a^{\mathcal{C}}}$ is the Dirac distribution at $1_a^{\mathcal{C}}: a \to a$ in $\mathcal{C}$. To see this acts as an identity, consider

$$\begin{tikzcd}
& !\mathcal{C}(b,b)\otimes\,!\mathcal{C}(a,b) \arrow[rr, "m^{!\mathcal{C}}"] \arrow[dr, "d\otimes d"] & & !\mathcal{C}(a,b) \arrow[d, "d"] \\
k\otimes !\mathcal{C}(a,b) \arrow[ur, "|\phi\rangle_{1_b}\otimes 1"] \arrow[rr, "1_b\otimes d"'] & & \mathcal{C}(b,b)\otimes\mathcal{C}(a,b) \arrow[r, "-\circ-"'] & \mathcal{C}(a,b)
\end{tikzcd}$$

To show $m^{!\mathcal{C}} \circ (|\phi\rangle_{1_b} \otimes 1) = 1_{!\mathcal{C}(a,b)}$ observe the LHS is a coalgebra morphism, so it suffices to check commutativity after applying d. But then it is clear. Similarly

$$\begin{tikzcd}
 & !\mathcal{C}(a,b) \otimes !\mathcal{C}(a,a) \arrow[r, "m^{!\mathcal{C}}"] \arrow[d, "d \otimes d"] & !\mathcal{C}(a,b) \arrow[d, "d"] \\
!\mathcal{C}(a,b) \otimes k \arrow[ur, "1 \otimes |\phi\rangle_{1_a}"] \arrow[r, "d \otimes 1_a"'] & \mathcal{C}(a,b) \otimes \mathcal{C}(a,a) \arrow[r, "-\circ-"'] & \mathcal{C}(a,b)
\end{tikzcd}$$

commutes. $\square$

<u>Remark</u> Note that $!\mathcal{C}$ is not just k-linear, but actually enriched over the category of coalgebras in k (Homs are coalgebras, composition maps are morphisms of coalgebras), which is symmetric monoidal (actually cartesian) under $\otimes k$. Here enrichment of $\mathcal{X}$ over Coalg_k means in particular units are given by coalgebra morphisms $k \longrightarrow \mathcal{X}(x,x)$, so 1_x satisfies $\Delta(1_x) = 1_x \otimes 1_x$.

<u>Lemma</u> There is a k-linear functor $F : !\mathcal{C} \longrightarrow \mathcal{C}$ defined by $F(x) = x$ for objects x, and

$$F_{a,b} : !\mathcal{C}(a,b) \longrightarrow \mathcal{C}(a,b) \quad \text{is} \quad F_{a,b} := d, \text{ the universal map.}$$

<u>Proof</u> This is clear from (2.1). $\square$

<u>Remark</u> One should view $!\mathcal{C}$ as the category of finitely supported distributions on the morphisms of $\mathcal{C}$ [2, §A.2], i.e. over each $f : a \to b$ sits the Dirac distribution $|\phi\rangle_f \in (!\mathcal{C})(a,b)$ and its derivatives $|\alpha_1, \dots, \alpha_r\rangle_f$

<u>Remark</u> Using the description of d from [1] we have $F_{a,b}(|\phi\rangle_f) = f$ and $F_{a,b}(|\alpha\rangle_f) = \alpha$, while $F_{a,b}(|\alpha_1, \dots, \alpha_r\rangle_f) = 0$ for $r \geq 2$.

<u>Remark</u> Using [1, Thm 2.22] and [2, §2.1] we can describe the composition

$$m^{!\mathscr{C}}_{a,b,c} : \; !\mathscr{C}(b,c) \otimes !\mathscr{C}(a,b) \longrightarrow \; !\mathscr{C}(a,c)$$

explicitly. Let $\alpha, \alpha_1, \dots, \alpha_r \in \mathscr{C}(b,c)$ and $\beta, \beta_1, \dots, \beta_s \in \mathscr{C}(a,b)$ be given, with $r, s \geq 0$. Then we first consider the map of (2.1), in the $\urcorner\downarrow$ direction, i.e.

$$\pi : \; !\big(\mathscr{C}(b,c) \oplus \mathscr{C}(a,b)\big) \cong \; !\mathscr{C}(b,c) \otimes !\mathscr{C}(a,b) \longrightarrow \mathscr{C}(a,c)$$

given by the formula

$$\begin{aligned} |\alpha_1, \dots, \alpha_r, \beta_1, \dots, \beta_s\rangle_{(\alpha,\beta)} \longmapsto\; & d(|\alpha_1, \dots, \alpha_r\rangle_\alpha) \circ d(|\beta_1, \dots, \beta_s\rangle_\beta) \\ =\; & \delta_{r=0}\delta_{s=0}\, \alpha\beta \\ +\; & \delta_{r=1}\delta_{s=0}\, \alpha_1\beta \\ +\; & \delta_{r=0}\delta_{s=1}\, \alpha\beta_1 \\ +\; & \delta_{r=1}\delta_{s=1}\, \alpha_1\beta_1 . \end{aligned} \tag{4.1}$$

Writing $\alpha_0 = \alpha$, $\beta_0 = \beta$ we can write this as $\delta_{r\leq 1}\delta_{s\leq 1}\alpha_r\beta_s$. The lifting is therefore the morphism of coalgebras [1, Thm 2.22]

$$\begin{aligned} !\big(\mathscr{C}(b,c) \oplus \mathscr{C}(a,b)\big) &\longrightarrow \; !\mathscr{C}(a,c) \\ |\alpha_1, \dots, \alpha_r, \beta_1, \dots, \beta_s\rangle_{(\alpha,\beta)} &\longmapsto \sum_{\substack{C \text{ a partition} \\ \text{of } \{1,\dots,r+s\}}} \big|\pi|C_1\rangle_{(\alpha,\beta)}, \dots, \pi|C_q\rangle_{(\alpha,\beta)}\big\rangle_{\alpha\beta} \end{aligned} \tag{4.2}$$

where $C \subseteq \{1, \dots, r, r+1, \dots, r+s\}$ stands for the product of the appropriate matching elements of $\alpha_1, \dots, \alpha_r, \beta_1, \dots, \beta_{r+s}$ in $\mathrm{Sym}(\mathscr{C}(b,c) \oplus \mathscr{C}(a,b))$. From (4.1) we deduce that the partition C is made up of subsets of size 1 or 2.

The description becomes easier when $\alpha = \beta = 0$ since then the only nonzero summands in (4.2) come from $r = s = 1$ in (4.1), and hence partitions C in which every element has size two, with one element from $\alpha_1, \dots, \alpha_r$ and one element from $\beta_1, \dots, \beta_s$. In particular:

Lemma We have

(i) $m^{!\mathcal{C}}_{a,b,c}(|\alpha_1, \dots, \alpha_r\rangle_0 \otimes |\beta_1, \dots, \beta_s\rangle_0) = 0$ unless $r = s \geq 1$

(ii) For $r \geq 1$, we have

$$m^{!\mathcal{C}}_{a,b,c}\big(|\alpha_1, \dots, \alpha_r\rangle_0 \otimes |\beta_1, \dots, \beta_r\rangle_0 \big) = \sum_{\gamma \in S_r} | \alpha_1 \beta_{\gamma(1)}, \dots, \alpha_r \beta_{\gamma(r)} \rangle_0. \qquad (5.1)$$

Observe that there is a commutative diagram

$$\begin{array}{ccc} !\mathcal{C}(b,c) \otimes !\mathcal{C}(a,b) & \xrightarrow{m^{!\mathcal{C}}} & !\mathcal{C}(a,c) \\ \cup\!| & & \cup\!| \\ \mathrm{Sym}_0(b,c) \otimes \mathrm{Sym}_0(a,b) & & \mathrm{Sym}_0(a,c) \\ \cup\!| & & \cup\!| \\ \mathrm{Sym}_0^r(b,c) \otimes \mathrm{Sym}_0^r(a,b) & \xrightarrow{Q} & \mathrm{Sym}_0^r(a,c) \end{array}$$

where

$$Q(\alpha_1 \otimes \cdots \otimes \alpha_r, \beta_1 \otimes \cdots \otimes \beta_r) = \sum_{\gamma \in S_r} (\alpha_1 \beta_{\gamma(1)}) \otimes \cdots \otimes (\alpha_r \beta_{\gamma(r)}) \qquad (5.2)$$

Def^n Let $Z^d_{\mathcal{C}} \subseteq !\mathcal{C}$ denote the category with the same objects as $\mathcal{C}$, and morphisms $(d \geq 1)$

$$Z^d_{\mathcal{C}}(a,b) = \mathrm{Sym}^d_0(\mathcal{C}(a,b)) = \{ |\alpha_1, \dots, \alpha_d\rangle_0 \mid \alpha_i \in \mathcal{C}(a,b) \} \subseteq (!\mathcal{C})(a,b).$$

The composition in $Z_{\mathscr{C}}^{d}$ is given by the composition in $!\mathscr{C}$ and thus by (5.1). The identity at $a \in ob(Z_{\mathscr{C}}^{d})$ is $\frac{1}{d!}|1_a, \ldots, 1_a\rangle_0$, as is clear from (5.1). So $Z_{\mathscr{C}}^{d}$ is a category but <u>not</u> a subcategory of $!\mathscr{C}$.

<u>Lemma</u> Let $\mathcal{H}, \mathscr{C}$ be k-linear categories with $\mathcal{H}$ enriched over $Coalg_k$, and $G : \mathcal{H} \longrightarrow \mathscr{C}$ a k-linear functor. Then there exists a functor $\hat{G}$ making

$$\begin{tikzcd} & !\mathscr{C} \arrow[d, "d"] \\ \mathcal{H} \arrow[ur, dashed, "\hat{G}"] \arrow[r, "G"'] & \mathscr{C} \end{tikzcd} \qquad (6.1)$$

commute (on the nose). Moreover $\hat{G}$ is a functor of $Coalg_k$-enriched categories, in the sense that on Hom-spaces it is compatible with the coalgebra structure.

<u>Proof</u> Define $\hat{G}$ on objects by $\hat{G}(x) = G(x)$. For $x, y \in \mathcal{H}$ consider

$$\begin{tikzcd} & !\mathscr{C}(Gx, Gy) \arrow[d, "d"] \\ \mathcal{H}(x,y) \arrow[ur, dashed, "\hat{G}_{x,y}"] \arrow[r, "G_{x,y}"'] & \mathscr{C}(Gx, Gy). \end{tikzcd}$$

We define $\hat{G}_{x,y}$ as the unique morphism of coalgebras making this diagram commute. We need to check that $\hat{G}$ is a k-linear functor. It is clear (4.1) commutes. Regarding identities, for $x = y$, we have

$$d(\hat{G}_{x,x}(1_x)) = G_{x,x}(1_x) = 1_{Gx}$$

Moreover since $1_x \in \mathcal{H}(x,x)$ is group-like, so is $\hat{G}_{x,x}(1_x)$, whence it is $|\emptyset\rangle_{1_{Gx}}$ as claimed. $\square$

(2) Strict polynomial functors

A strict polynomial functor $\mathrm{mod}\,k \xrightarrow{G} \mathrm{mod}\,k$ is a functor whose action on morphism spaces is computed by polynomial maps, i.e. morphisms of affine k-schemes

$$\mathrm{Hom}_k(V,W) \longrightarrow \mathrm{Hom}_k(GV, GW).$$

The reference is [10], see also [6,9]. Before giving the precise definition, recall that for k-vector spaces V, W the associated schemes are $\mathrm{Spec}(\mathrm{Sym}(V^*))$, $\mathrm{Spec}(\mathrm{Sym}(W^*))$, so

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Sch}/k}(V,W) &:= \mathrm{Hom}_{\mathrm{Sch}/k}(\mathrm{Spec}(\mathrm{Sym}(V^*)), \mathrm{Spec}(\mathrm{Sym}(W^*))) \\ &\cong \mathrm{Hom}_{\mathrm{Alg}/k}(\mathrm{Sym}(W^*), \mathrm{Sym}(V^*)) \\ &\cong \mathrm{Hom}_k(W^*, \mathrm{Sym}(V^*)) \\ &\cong W \otimes \mathrm{Sym}(V^*). \end{aligned}$$

We say a function $V \xrightarrow{g} W$ is <u>polynomial</u> (resp. <u>polynomial of degree d</u>) if there is a tensor $\sum_i w_i \otimes q_i$ with $q_i \in \mathrm{Sym}(V^*)$ (resp. all $q_i \in \mathrm{Sym}^d(V^*)$) such that for all $\mu \in W^*$ we have as functions $V \longrightarrow k$ an equality

$$\mu \circ g = \textstyle\sum_i \mu(w_i) \cdot q_i.$$

<u>Def^n</u> A functor $G : \mathrm{mod}\,k \longrightarrow \mathrm{mod}\,k$ is <u>strict polynomial</u> if (note G is <u>not</u> assumed linear!)

(i) $\forall V, W \in \mathrm{mod}\,k$ the function $G_{V,W} : \mathrm{Hom}_k(V,W) \longrightarrow \mathrm{Hom}_k(GV, GW)$ is polynomial in the sense just defined, and

(ii) there is a constant N such that $G_{V,V} : \mathrm{End}_k(V) \longrightarrow \mathrm{End}_k(GV)$ has degree $\leq N$ for all $V \in \mathrm{mod}\,k$, (i.e. these $G_{V,V}$ have a representation using q_i of total deg $\leq N$).

We say G is <u>homogeneous of degree d</u> if $G_{V,W}$ is so, for all V, W.

<u>Def^n</u> Let $\mathcal{P} = \mathcal{P}(\mathrm{mod}_k, \mathrm{mod}_k)$ denote the category of strict polynomial functors and natural transformations, and $\mathcal{P}_d$ the full subcategory of functors homogeneous of degree d.

<u>Lemma</u> $\mathcal{P} \cong \bigoplus_{d \geq 0} \mathcal{P}_d$, i.e. every functor decomposes uniquely as a direct sum of finitely many homogeneous functors.

<u>Remark</u> Given $G \in \mathcal{P}$ for $V, W \in \mathrm{mod}_k$ we have a tensor

$$G_{V,W} \in \mathrm{Hom}_k(GV, GW) \otimes \mathrm{Sym}(\mathrm{Hom}_k(V,W)^*).$$

Let us see how to express that G is a functor purely in terms of these tensors.

- Given U, V, W commutativity of (scheme maps)

$$\begin{tikzcd}
\mathrm{Hom}_k(V,W) \times \mathrm{Hom}_k(U,V) \arrow[r, "G_{V,W} \times G_{U,V}"] \arrow[d, "-\circ-"'] & \mathrm{Hom}_k(GV,GW) \times \mathrm{Hom}_k(GU,GV) \arrow[d, "-\circ-"] \\
\mathrm{Hom}_k(U,W) \arrow[r, "G_{U,W}"'] & \mathrm{Hom}_k(GU,GW)
\end{tikzcd}$$

means commutativity of algebra maps

$$\begin{tikzcd}
\mathrm{Sym}(\mathrm{Hom}_k(V,W)^*) \otimes \mathrm{Sym}(\mathrm{Hom}_k(U,V)^*) & \mathrm{Sym}(\mathrm{Hom}_k(GV,GW)^*) \otimes \mathrm{Sym}(\mathrm{Hom}_k(GU,GV)^*) \arrow[l, "G_{V,W} \otimes G_{U,V}"'] \\
\mathrm{Sym}_k(\mathrm{Hom}_k(U,W)^*) \arrow[u, "c_1"] & \mathrm{Sym}(\mathrm{Hom}_k(GU,GW)^*) \arrow[l, "G_{U,W}"'] \arrow[u, "c_2"']
\end{tikzcd}$$

where c_1 is induced by the dual of the composition map

$$\mathrm{Hom}_k(U,W)^* \longrightarrow [\mathrm{Hom}_k(V,W) \otimes \mathrm{Hom}_k(U,V)]^* \cong \mathrm{Hom}_k(V,W)^* \otimes \mathrm{Hom}_k(U,V)^*$$

and similarly c_2.

- To say $G(1_V) = 1_{GV}$ is to say that the k-point $1_V : \mathrm{Sym}(\mathrm{Hom}_k(V,V)^*) \longrightarrow k$ sends

$$G_{V,V} \in \mathrm{Hom}_k(GV,GV) \otimes \mathrm{Sym}(\mathrm{Hom}_k(V,V)^*),$$

to 1_{GV}. That is, noting that $h \in \mathrm{Sym}(T^*)$ may be <u>evaluated</u> at $t \in T$, if $G_{V,V} = \sum_i w_i \otimes q_i$ we require that $\sum_i w_i \cdot q_i(1_V) = 1_{GV}$.

If $G \in \mathcal{P}_d$, then for $V, W \in \mathrm{mod}_k$ we have a polynomial map of degree d

$$G_{V,W} : \mathrm{Hom}_k(V,W) \longrightarrow \mathrm{Hom}_k(GV, GW)$$

and hence tensor in

$$\begin{aligned} &\mathrm{Hom}_k(GV,GW) \otimes \mathrm{Sym}^d(\mathrm{Hom}_k(V,W)^*) \\ &\cong T^d(\mathrm{Hom}_k(V,W))^* \otimes \mathrm{Hom}_k(GV,GW) \qquad (9.2) \\ &\cong \mathrm{Hom}_k(T^d \mathrm{Hom}_k(V,W), \mathrm{Hom}_k(GV,GW)), \end{aligned}$$

where the <u>divided powers</u> are given by $T^d V = (V^{\otimes d})^{S_d}$. In characteristic zero the map

$$\begin{tikzcd} T^d V \arrow[r, shift left, "a_1\otimes\cdots\otimes a_d \,\mapsto\, \frac{1}{d!} a_1 \otimes \cdots \otimes a_d"] & \mathrm{Sym}^d V \arrow[l, shift left, "\sum_{\sigma\in S_d} a_{\sigma 1}\otimes\cdots\otimes a_{\sigma d} \,\mapsfrom\, a_1\otimes\cdots\otimes a_d"] \end{tikzcd} \qquad (9.3)$$

is an isomorphism. In general $\mathrm{Sym}^d(V)^* \cong T^d(V^*)$.

The discussion above shows that these associated linear maps

$$\widetilde{G}_{V,W} : T^d \mathrm{Hom}_k(V,W) \longrightarrow \mathrm{Hom}_k(GV, GW) \qquad (10.1)$$

send, in the case $V = W$, $1_V^{\otimes d}$ to 1_{GV}, and that they are functorial with respect to the following composition:

<u>Defn</u> We define a k-linear category $T^d \mathrm{mod}_k$

$$\mathrm{ob}(T^d \mathrm{mod}_k) = \mathrm{ob}(\mathrm{mod}_k)$$
$$(T^d \mathrm{mod}_k)(V, W) = T^d \mathrm{Hom}_k(V, W).$$

Composition is defined to be the map m induced in

$$\begin{tikzcd}
\left[\mathrm{Hom}_k(V,W)^{\otimes d}\right]^{S_d} \otimes \left[\mathrm{Hom}_k(U,V)^{\otimes d}\right]^{S_d} \arrow[r, phantom, "\subseteq"] \arrow[dd, dashed, "m"'] & \mathrm{Hom}_k(V,W)^{\otimes d} \otimes \mathrm{Hom}_k(U,V)^{\otimes d} \arrow[d, "\cong\ \text{(any choice of pairing)}"] \\
 & \left[\mathrm{Hom}_k(V,W) \otimes \mathrm{Hom}_k(U,V)\right]^{\otimes d} \arrow[d, "\mathrm{comp}^{\otimes d}"] \\
\left[\mathrm{Hom}_k(U,W)^{\otimes d}\right]^{S_d} \arrow[r, hook] & \mathrm{Hom}_k(U,W)^{\otimes d}
\end{tikzcd}$$

The identities are $1_V^{T^d \mathrm{mod}_k} = 1_V^{\otimes d}$.

The upshot of the above is

<u>Lemma</u> There are equivalences

(i) $\mathcal{P}_d \cong [T^d \mathrm{mod}_k, \mathrm{mod}_k]_k$

(ii) $\mathcal{P} \cong \bigoplus_{d \geq 0} [T^d \mathrm{mod}_k, \mathrm{mod}_k]_k$

Def^n (notation of [6]) We write Γ^d_k for $\Gamma^d \mathrm{mod}k$ and $\mathrm{rep}\,\Gamma^d_k$ for $[\Gamma^d \mathrm{mod}k, \mathrm{mod}k]_k$

Note Similarly we can define strict polynomial functors $\mathcal{C} \to \mathcal{D}$, $\mathcal{C}, \mathcal{D}$ enriched over mod$k$.

3 The connection

We have discussed the "free" enrichment $!\mathcal{C}$ of a k-linear category $\mathcal{C}$ over coalgebras, and the category $\mathcal{P}_d \cong \mathrm{rep}\,\Gamma^d_k$ of strict polynomial functors of degree d. Since the spaces $\mathrm{Sym}^d(\mathcal{C}(a,b)) \cong \Gamma^d\mathcal{C}(a,b)$ appear in $!\mathcal{C}$ the existence of a connection between $[!\mathrm{mod}k, \mathrm{mod}k]_k$ and $\mathcal{P}_d$ is obvious; we now spell it out.

Recall the category $Z^d_{\mathcal{C}}$ of p. 5, consisting of morphisms in $!\mathcal{C}$ composed according to the composition rule of $!\mathcal{C}$, but without the identity of $!\mathcal{C}$.

Lemma With $\mathcal{C} = \mathrm{mod}k$ there is an equivalence $Z^d_{\mathcal{C}} \cong \Gamma^d_k$.

Proof We have $\mathrm{ob}(Z^d_{\mathcal{C}}) = \mathrm{ob}(\mathrm{mod}k)$ and for vector spaces V, W we have

$$\begin{aligned} Z^d_{\mathcal{C}}(V,W) &= \mathrm{Sym}^d(\mathrm{Hom}_k(V,W)) \\ &\cong \Gamma^d(\mathrm{Hom}_k(V,W)). \end{aligned} \tag{11.1}$$

From (5.1) we see that the maps

$$\begin{aligned} \mathrm{Sym}^d\,\mathrm{Hom}_k(V,W) &\longrightarrow \Gamma^d\,\mathrm{Hom}_k(V,W) \\ \alpha_1 \otimes \cdots \otimes \alpha_d &\longmapsto \sum_{\sigma \in S_d} \alpha_{\sigma(1)} \otimes \cdots \otimes \alpha_{\sigma(d)} \end{aligned} \tag{11.2}$$

give an equivalence $Z^d_{\mathcal{C}} \xrightarrow{\cong} \Gamma^d_{\mathcal{C}}$. $\square$

<u>Lemma</u> There is a fully faithful functor $\mathrm{rep}\,T^d_k \xhookrightarrow{\Phi} [\,!\mathrm{mod}k, \mathrm{mod}k\,]_k$.

<u>Proof</u> Let $F \in \mathrm{rep}\,T^d_k$ be given. We define $\Phi(F)$ on objects by $\Phi(F)(V) = F(V)$. On morphisms we define, for $V, W \in \mathrm{mod}k$,

$$\Phi(F)_{V,W} : (!\mathrm{mod}k)(V,W) = \bigoplus_{f : V \to W} \mathrm{Sym}_f(\mathrm{Hom}_k(V,W)) \longrightarrow \mathrm{Hom}_k(FV, FW)$$

to restrict to zero on $\mathrm{Sym}_f(\mathrm{Hom}_k(V,W))$, unless $f = 0$ <u>or</u> $V = W$, $f = 1_V$. These cases only overlap when $V \cong 0$, in which case we have a third prescription:

<u>Case $f = 0$</u>: $\Phi(F)_{V,W}|_{\mathrm{Sym}_0}$ vanishes on Sym^i_0 unless $i = d$. In this case it is defined using (11.2) to be

$$\begin{array}{c}\Phi(F)_{V,W}|_{\mathrm{Sym}^d_0} = \mathrm{Sym}^d\mathrm{Hom}_k(V,W) \xrightarrow[\cong]{} T^d\mathrm{Hom}_k(V,W) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow F_{V,W} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \mathrm{Hom}_k(FV,FW)\end{array}$$

<u>Case $V = W$, $f = 1_V$</u> $(V \not\cong 0)$ $\Phi(F)_{V,V}|_{\mathrm{Sym}_1}$ vanishes on Sym^i_1 unless $i = 0$, and in this case it is defined by

$$\Phi(F)_{V,V}|_{\mathrm{Sym}^0_1} : k \cdot |\emptyset\rangle_{1_V} \longrightarrow \mathrm{Hom}_k(FV, FV)$$

sending $|\emptyset\rangle_{1_V}$ to 1_{FV}.

It is clear from the previous page that $\Phi(F)$ is a k-linear functor.

<u>Case $V = W \cong 0$, $f = 1_V = 0$</u>, $\Phi(F)_{V,V}|_{\mathrm{Sym}^i_0}$ is nonvanishing for $i = 0$ where it is given by the above, and since $\mathrm{Sym}^i(0) \cong 0$ for $i > 0$ that is all there is to say.

It is clear from p. 11 that, thus defined, $\Phi(F)$ is a k-linear functor. Any natural transformation $\psi : F_1 \longrightarrow F_2$ gives $\Phi(\psi) : \Phi(F_1) \longrightarrow \Phi(F_2)$ defined by $\Phi(\psi)_V = \psi_V$, which is clearly natural. Thus Φ is a well-defined functor (k-linear), and it is easily checked to be full. □

This completes the connection between strict polynomial functors $\mathcal{P}_d \cong \operatorname{rep} T^d_k$ and the construction $\mathcal{C} \longmapsto !\mathcal{C}$ on k-linear categories. One could dress this up in the following way: let $\mathcal{B}$ be a bicategory of Modk-enriched categories, $\mathcal{A}$ a bicategory of Coalgk-enriched categories. The existence of an adjunction

$$\mathrm{Coalg}_k \underset{F_\rho}{\overset{F}{\rightleftarrows}} \mathrm{Mod}_k \qquad F \dashv F_\rho \qquad (\text{so } ! = F \circ F_\rho)$$

where F is the forgetful functor, gives rise to an adjunction

$$\mathcal{A} \underset{G_\rho}{\overset{G}{\rightleftarrows}} \mathcal{B}$$

where G is also forgetful, and $G_\rho(\mathcal{C}) = !\mathcal{C}$. In the Kleisli bicategory $\mathcal{K}_!$ for the comonad $! = G \circ G_\rho$ on $\mathcal{B}$ (called $\mathcal{V}_k$ in the introduction) we have

$$\mathrm{Hom}_{\mathcal{K}_!}(\mathcal{C}, \mathcal{C}') = \mathrm{Hom}_{\mathcal{B}}(!\mathcal{C}, \mathcal{C}') = [\,!\mathcal{C}, \mathcal{C}'\,]_k .$$

<u>Upshot</u> The bicategory of <u>k-linear categories</u> and <u>strict polynomial functors</u> is a sub-bicategory of the Kleisli bicategory of the comonad ! on the bicategory $\mathcal{V}_k$ of k-linear categories and k-linear functors.

Since linear logic is precisely the language expressing all formal constructions possible with $\otimes_k$, $[-,-]_k$ and ! using the various units and counits of adjunction and the functors themselves, this suggests a natural connection to polynomial functors.

I have not checked the above in any detail, but the only nontrivial point seems to be that composition in the Kleisli bicategory matches composition of polynomial functors. To see this, let $X, Y : \mathrm{mod}_k \longrightarrow \mathrm{mod}_k$ be strict polynomial of degrees d, e. Then it is easy to see $Y \circ X$ is strict polynomial of degree de (using the def^n of p. 7). Let

$$\tilde{X} : T^d \mathrm{mod}_k \longrightarrow \mathrm{mod}_k, \quad \tilde{Y} : T^e \mathrm{mod}_k \longrightarrow \mathrm{mod}_k \tag{14.1}$$

be the associated <u>linear</u> functors, and $\Phi(\tilde{X}), \Phi(\tilde{Y}) \in [\,!\mathrm{mod}_k, \mathrm{mod}_k]_k$ as defined on p. 12. These are both arrows $\mathrm{mod}_k \longrightarrow \mathrm{mod}_k$ in the Kleisli category, and their composite is

$$!\mathrm{mod}_k \longrightarrow\ !!\mathrm{mod}_k \xrightarrow{\ !\Phi(\tilde{X})\ }\ !\mathrm{mod}_k \xrightarrow{\ \Phi(\tilde{Y})\ } \mathrm{mod}_k .$$

By the universal property we can also compute this by first <u>lifting</u> $\Phi(\tilde{X})$ to $!\mathrm{mod}_k \xrightarrow{\Phi(\tilde{X})_{\mathrm{lift}}} !\mathrm{mod}_k$ (a functor compatible with the coalgebra structures) and <u>then</u> composing, i.e.

$$\begin{tikzcd}
!\mathrm{mod}_k \arrow[r, "\Phi(\tilde{X})"] \arrow[dr, dashed, "\Phi(\tilde{X})_{\mathrm{lift}}"'] & \mathrm{mod}_k & \mathrm{mod}_k \\
& !\mathrm{mod}_k \arrow[u] \arrow[ur, "\Phi(\tilde{Y})"'] &
\end{tikzcd} \tag{14.2}$$

On objects this is $V \longmapsto Y(X(V))$, and we may compute it on morphisms using [1, Prop 2.21], which says that for $V, W \in \mathrm{mod}_k$ and $\alpha_1, \dots, \alpha_s, f \in \mathrm{Hom}_k(V, W)$

$$\begin{array}{cccc}
\Phi(\tilde{X})_{\mathrm{lift}, V, W} : & (!\mathrm{mod}_k)(V, W) & \longrightarrow & (!\mathrm{mod}_k)(XV, XW) \\
& \| & & \| \\
& \bigoplus_{f : V \to W} \mathrm{Sym}(\mathrm{Hom}_k(V, W)) & & \bigoplus_{g : XV \to XW} \mathrm{Sym}(\mathrm{Hom}_k(V, W))
\end{array} \tag{14.3}$$

is given by (15.1)

$$\Phi(\tilde{X})_{\mathrm{lift},V,W}\left(|\alpha_1,\dots,\alpha_s\rangle_f\right) = \sum_{\substack{\text{partitions}\\ C \text{ of } \{1,\dots,s\}}} \Big| \Phi(\tilde{X})_{V,W}|\alpha_{C_1}\rangle_f, \dots, \Phi(\tilde{X})_{V,W}|\alpha_{C_q}\rangle_f \Big\rangle_g$$

where $g = \Phi(\tilde{X})_{V,W}|\phi\rangle_f$, and α_{C_i} stands for the subtensor of $\alpha_1 \otimes \cdots \otimes \alpha_s$ picked out by the element C_i of the partition $C = \{C_1, \dots, C_q\}$. Now the way $\Phi(\tilde{X})$ is defined this is zero unless $f = 0$ or $V = W$ and $f = 1_V$.

<u>Case $V=W$, $f = 1_V$</u> (15.1) is zero unless $s = 0$, where it gives $\Phi(\tilde{X})_{\mathrm{lift},V,V}|\phi\rangle_{1_V}$ is equal to $|\phi\rangle_{1_{XV}}$, which is just part of $\Phi(\tilde{X})_{\mathrm{lift}}$ being a functor.

<u>Case $V \neq W$, $f = 0$</u> the right hand of (15.1) is zero unless $|C_i| = d$ for all i, which means $s = ad$ for some $a \geq 1$. Moreover $g = 0$, and if $|C_i| = d$ then

$$\Phi(\tilde{X})_{V,W}|\alpha_{C_i}\rangle_0 = \tilde{X}_{V,W}\Big(\sum_{\delta \in S_d} \delta(\alpha_{C_i})\Big)$$

where $\tilde{X}_{V,W} : T^d \operatorname{Hom}_k(V,W) \longrightarrow \operatorname{Hom}_k(XV, XW)$, and δ acts on $\operatorname{Hom}_k(V,W)^{\otimes d}$ in the obvious way. In particular the RHS of (15.1) belongs to $\operatorname{Sym}^a_0 \operatorname{Hom}_k(XV, XW)$.

Referring to (14.2) we conclude that for $V \neq W$,

- $\left[\Phi(\tilde{Y}) \circ \Phi(\tilde{X})_{\mathrm{lift}}\right]_{V,W}$ is nonzero only on $\operatorname{Sym}^{de}_0 \operatorname{Hom}_k(V,W)$
- on $\alpha_1, \dots, \alpha_{de} \in \operatorname{Hom}_k(V,W)$ it takes the value given overleaf

$$\Phi(\tilde{Y})_{XV,XW}\Big(\Phi(\tilde{X})_{\mathrm{lift},V,W}\langle\alpha_1,\dots,\alpha_{de}\rangle_0\Big) = \sum_{\substack{\text{partitions}\\ C \text{ of } \{1,\dots,de\}\\ \text{with each}\\ |C_i| = d}} \Phi(\tilde{Y})_{XV,XW}\Big\langle \tilde{X}_{V,W}\Big(\sum_\sigma \sigma(\alpha_{C_1})\Big),\dots,\tilde{X}_{V,W}\Big(\sum_\sigma \sigma(\alpha_{C_e})\Big)\Big\rangle_0 \tag{16.1}$$

$$= \sum_{\substack{\text{partition } C\\ \text{as above}}} \tilde{Y}_{XV,XW}\Big(\sum_{\tau\in S_e} \tilde{X}_{V,W}\Big(\sum_\sigma \sigma(\alpha_{C_{\tau(1)}})\Big)\otimes\cdots\otimes \tilde{X}_{V,W}\Big(\sum_\sigma \sigma(\alpha_{C_{\tau(e)}})\Big)\Big)$$

We compare this to the morphism of schemes

$$\mathrm{Hom}_k(V,W) \xrightarrow{X_{V,W}} \mathrm{Hom}_k(XV,XW) \xrightarrow{Y_{XV,XW}} \mathrm{Hom}_k(YXV,YXW)$$

computed by contracting

$$\tilde{X}_{V,W} \in \mathrm{Hom}_k(XV,XW)\otimes \mathrm{Sym}^d(\mathrm{Hom}_k(V,W)^*) \qquad \tilde{Y}_{XV,XW} \in \mathrm{Hom}_k(YXV,YXW)\otimes \mathrm{Sym}^e(\mathrm{Hom}_k(XV,XW)^*)$$
$$\cong T^d\,\mathrm{Hom}_k(V,W)^*\otimes \mathrm{Hom}_k(XV,XW)$$

along the shared degrees of freedom, i.e. evaluation of the polynomial in functionals part of $\tilde{Y}$ on the points in $\mathrm{Hom}_k(XV,XW)$ of $\tilde{X}$. Modulo some combinatorics this seems likely to agree with (16.1) (TODO)

This justifies the "Upshot" claim on p. 13, i.e.

<u>Lemma</u> Given strict polynomial functors $X, Y : \mathrm{mod}_k \longrightarrow \mathrm{mod}_k$ with associated k-linear functors $\Phi(\tilde{X}), \Phi(\tilde{Y}) : !\mathrm{mod}_k \longrightarrow \mathrm{mod}_k$ we have $\Phi(\widetilde{YX})$ equal to

$$!\mathrm{mod}_k \longrightarrow !!\mathrm{mod}_k \xrightarrow{!\Phi(\tilde{X})} !\mathrm{mod}_k \xrightarrow{\Phi(\tilde{Y})} \mathrm{mod}_k.$$

which is also equal to $\Phi(\tilde{Y})\circ\Phi(\tilde{X})_{\mathrm{lift}}$.

4 Linear logic with polynomial functors

To summarise the discussion thus far: a strict polynomial functor of degree d

$$X : \text{mod}_k \longrightarrow \text{mod}_k \qquad (17.1)$$

is the same data as a k-linear functor (see (10.1))

$$\tilde{X} : T^d \text{mod}_k \longrightarrow \text{mod}_k \qquad (17.2)$$

and these in turn may be identified with a special class of k-linear functors

$$\Phi(\tilde{X}) : !\text{mod}_k \longrightarrow \text{mod}_k, \qquad (17.3)$$

by the Lemma on p. 12. Moreover the composition $Y \circ X$, for $Y : \text{mod}_k \longrightarrow \text{mod}_k$ polynomial of degree e, may be realised as composition of $\Phi(\tilde{X})$, $\Phi(\tilde{Y})$ using the structure of the comonad ! on k-linear categories (p. 13 - 16).

We now explain how linear logic provides algorithms for constructing new functors out of input data like $\Phi(\tilde{X})$. There are a few final ingredients that need to be introduced first, however:

- Comultiplication on $!\mathcal{C}$: let $\mathcal{C}$ be k-linear, and define a k-linear functor

$$\Delta : !\mathcal{C} \longrightarrow !\mathcal{C} \otimes !\mathcal{C}$$

to be the diagonal on objects, $\Delta(a) = (a,a)$ and on morphisms to be the comultiplication in the cofree coalgebra $!\mathcal{C}(a,b)$, i.e.

$$\Delta_{a,b} = \Delta_{!\mathcal{C}(a,b)} : \; !\mathcal{C}(a,b) \longrightarrow (!\mathcal{C}\otimes !\mathcal{C})\big((a,a),(b,b)\big) = \; !\mathcal{C}(a,b) \otimes !\mathcal{C}(a,b).$$

To see this is a functor we need to check commutativity of

$$\begin{tikzcd}
!\mathcal{C}(b,c)\otimes !\mathcal{C}(a,b) \arrow[rr, "-\circ-"] \arrow[d, "\Delta_{b,c}\otimes\Delta_{a,b}"] & & !\mathcal{C}(a,c) \arrow[d, "\Delta_{a,c}"] \\
\big(!\mathcal{C}(b,c)\otimes !\mathcal{C}(b,c)\big)\otimes\big(!\mathcal{C}(a,b)\otimes !\mathcal{C}(a,b)\big) \arrow[d, phantom, "\wr\wr"] & & !\mathcal{C}(a,c)\otimes !\mathcal{C}(a,c) \\
!\mathcal{C}(b,c)\otimes !\mathcal{C}(a,b)\otimes !\mathcal{C}(b,c)\otimes !\mathcal{C}(a,b) \arrow[urr, bend right, "(-\circ-)\otimes(-\circ-)"'] & &
\end{tikzcd}$$

which holds because by construction the composition $-\circ- = m^{!\mathcal{C}}_{a,b,c}$ is a morphism of coalgebras. Compatibility with identities holds since $|\phi\rangle_{1a}$ is group-like. Clearly this $\Delta : !\mathcal{C} \longrightarrow !\mathcal{C}\otimes !\mathcal{C}$ is coassociative in a suitable sense, and cocommutative.

- Counit on $!\mathcal{C}$ define a functor $c : !\mathcal{C} \longrightarrow k$ (k being the one-object category with $\mathrm{End}(\bullet) = k$) by sending every object to $\bullet$ and $c_{a,b} : !\mathcal{C}(a,b) \longrightarrow k$ the counit. Then the diagrams

$$\begin{tikzcd}
!\mathcal{C} \cong !\mathcal{C}\otimes k & !\mathcal{C}\otimes !\mathcal{C} \arrow[l, "1\otimes c"'] \arrow[r, "c\otimes 1"] & k\otimes !\mathcal{C} \cong !\mathcal{C} \\
& !\mathcal{C} \arrow[ul, equal] \arrow[u, "\Delta"] \arrow[ur, equal] &
\end{tikzcd}$$

commute, so $(!\mathcal{C}, \Delta, c)$ is a "coalgebra object" in k-linear categories.

With these ingredients in place, we can describe the class of constructions described by the language of linear logic. We will do this by juxtaposing categorical constructions with the deduction rules of linear logic (see [4] for more detail)

- Throughout $A, B, C, \ldots$ stand for formulas of linear logic, built from atomic formulas $x, y, z, \ldots$ using connectives $\otimes$, $\multimap$ and $!$ (e.g. $!(x \multimap x) \multimap (x \multimap x)$) where $\otimes, \multimap$ are binary and $!$ is unary.

- $\mathcal{A}, \mathcal{B}, \mathcal{C}$ stand for small k-linear categories (there are some set-theoretic issues since we want to write e.g. $[[\mathcal{A}, \mathcal{B}]_k, \mathcal{C}]_k$ but we ignore this for now), and $\pi, \rho, \kappa, \theta, \ldots$ for k-linear functors

	Categorical construction	Deduction Rule
Axiom Rule	$1_{\mathcal{A}} : \mathcal{A} \longrightarrow \mathcal{A}$	$\dfrac{}{A \vdash A}\,ax$
Right $\otimes$-rule	Given $\pi : \mathcal{A} \longrightarrow \mathcal{A}'$, $\rho : \mathcal{B} \longrightarrow \mathcal{B}'$ form the tensor product $\pi \otimes \rho : \mathcal{A} \otimes \mathcal{B} \longrightarrow \mathcal{A}' \otimes \mathcal{B}'$	$\dfrac{\overset{\pi}{\vdots} \atop A \vdash A' \qquad \overset{\rho}{\vdots} \atop B \vdash B'}{A, B \vdash A', B'}\,\otimes R$
Left $\otimes$-rule	Does nothing: tells us to interpret , on the LHS of $\vdash$ as a $\otimes$	$\dfrac{\overset{\pi}{\vdots} \atop A, B \vdash C}{A \otimes B \vdash C}\,\otimes L$
Right $\multimap$ rule	Given $\pi : \mathcal{A} \otimes \mathcal{B} \longrightarrow \mathcal{C}$ produces the adjoint functor $\mathcal{A} \longrightarrow [\mathcal{B}, \mathcal{C}]_k$	$\dfrac{\overset{\pi}{\vdots} \atop A, B \vdash C}{A \vdash B \multimap C}\,\multimap R$

Left ⊸ rule

Given $\pi : \mathcal{A} \longrightarrow \mathcal{B}$, $\rho : \mathcal{C} \longrightarrow \mathcal{D}$ produces the functor

$$\mathcal{A} \otimes [\mathcal{B}, \mathcal{C}]_k \longrightarrow \mathcal{D}$$
$$(a, F) \longmapsto \rho(F(\pi a))$$

$$\dfrac{\overset{\pi}{\vdots} \atop A \vdash B \qquad \overset{\rho}{\vdots} \atop C \vdash D}{A, B \multimap C \vdash D} \multimap L$$

Right ! rule (Promotion)

Given $\pi : !\mathcal{A} \longrightarrow \mathcal{B}$ uses the universal property (p. 6) to produce $\dashrightarrow$ in

$$\begin{tikzcd} !\mathcal{A} \arrow[r, "\pi"] \arrow[dr, dashed] & \mathcal{B} \\ & !\mathcal{B} \arrow[u, "d"'] \end{tikzcd}$$

$$\dfrac{\overset{\pi}{\vdots} \atop !A \vdash B}{!A \vdash !B} \text{ prom}$$

Left ! rule (Dereliction)

Given $\pi : \mathcal{A} \longrightarrow \mathcal{B}$ precomposes with the universal $d : !\mathcal{A} \longrightarrow \mathcal{A}$ to obtain $\pi \circ d : !\mathcal{A} \longrightarrow \mathcal{B}$.

$$\dfrac{\overset{\pi}{\vdots} \atop A \vdash B}{!A \vdash B} \text{ der}$$

Contraction

Given $\pi : !\mathcal{A} \otimes !\mathcal{A} \longrightarrow \mathcal{B}$ precomposes with $\Delta : !\mathcal{A} \longrightarrow !\mathcal{A} \otimes !\mathcal{A}$ to obtain $\pi \circ \Delta : !\mathcal{A} \longrightarrow \mathcal{B}$.

$$\dfrac{\overset{\pi}{\vdots} \atop !A, !A \vdash B}{!A \vdash B} \text{ ctr}$$

Weakening

Given $\pi : \mathcal{A} \longrightarrow \mathcal{B}$ precomposes with $c : !\mathcal{C} \longrightarrow k$ to obtain $!\mathcal{C} \otimes \mathcal{A} \longrightarrow k \otimes \mathcal{A} \cong \mathcal{A} \xrightarrow{\pi} \mathcal{B}$.

$$\dfrac{\overset{\pi}{\vdots} \atop A \vdash B}{!C, A \vdash B} \text{ weak}$$

Cut

Given $\pi : \mathcal{A} \longrightarrow \mathcal{B}$, $\rho : \mathcal{B} \longrightarrow \mathcal{C}$ form the composition $\rho \circ \pi : \mathcal{A} \longrightarrow \mathcal{C}$.

$$\dfrac{\overset{\pi}{\vdots} \atop A \vdash B \qquad \overset{\rho}{\vdots} \atop B \vdash C}{A \vdash C} \text{ cut}$$

Example 1 The following proof (= tree of deduction rules with all leaves Axiom Rules)

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}\,ax \qquad \dfrac{\dfrac{}{A\vdash A}\,ax \quad \dfrac{}{A \vdash A}\,ax}{A, A \multimap A \vdash A}\,\multimap L}{A, A\multimap A, A \multimap A \vdash A}\,\multimap L}{A\multimap A, A \multimap A \vdash A \multimap A}\,\multimap R}{!(A\multimap A), !(A \multimap A) \vdash A \multimap A}\,\text{der twice}}{!(A\multimap A) \vdash A \multimap A}\,\text{ctr} \tag{21.1}
$$

encodes the functor (21.2)

$$
![\mathcal{A},\mathcal{A}]_k \xrightarrow{\Delta} ![\mathcal{A},\mathcal{A}]_k \otimes ![\mathcal{A},\mathcal{A}]_k \xrightarrow{d \otimes d} [\mathcal{A},\mathcal{A}]_k \otimes [\mathcal{A},\mathcal{A}]_k \xrightarrow{comp} [\mathcal{A},\mathcal{A}]_k,
$$

which sends a linear functor $H \in [\mathcal{A},\mathcal{A}]_k$ (note that $H \in ob([\mathcal{A},\mathcal{A}]_k) = ob(![\mathcal{A},\mathcal{A}]_k)$) to

$$
H \longmapsto (H,H) \longmapsto (H,H) \longmapsto H \circ H. \tag{21.3}
$$

and on morphism spaces is

$$
\begin{tikzcd}
!Nat(H,H') \arrow[r, "\Delta"] & !Nat(H,H') \otimes !Nat(H,H') \arrow[r, "d \otimes d"] & Nat(H,H') \otimes Nat(H,H') \arrow[d, "\text{horizontal comp}"] \\
 & & Nat(H,H')
\end{tikzcd} \tag{21.4}
$$

where the last map is the horizontal composition

$$
\begin{tikzcd}
\mathcal{A} \arrow[r, bend left=50, "H"] \arrow[r, bend right=50, "H'"'] \arrow[r, phantom, "\Downarrow\phi"] & \mathcal{A} \arrow[r, bend left=50, "H"] \arrow[r, bend right=50, "H'"'] \arrow[r, phantom, "\Downarrow\psi"] & \mathcal{A}
\end{tikzcd}
\qquad
\begin{array}{c}
(\psi * \phi) : H \circ H \longrightarrow H' \circ H' \\
\| \\
(\psi H') \circ (H \phi)
\end{array}
$$

Given a natural transformation $f : H \Rightarrow H'$, evaluating (21.4) on $|\phi\rangle_f$ gives

$$|\phi\rangle_f \longmapsto |\phi\rangle_f \otimes |\phi\rangle_f \longmapsto f \otimes f \longmapsto f * f. \qquad (22.1)$$

So the semantics of (21.1), called the <u>Church numeral</u> $\underline{2}$, is to both square endofunctors and natural transformations (the latter via horizontal composition).

<u>Example 2</u> In the context of the previous example the link between the <u>proof</u> (21.1) and the <u>functor</u> (21.2) was made by choosing $\mathcal{A}$ as the denotation of A, written $[\![A]\!] = \mathcal{A}$ (one should think of the logical formulas as objects and terms/proofs as arrows, so that the language forms a freely generated category over the set of its atomic formulas. Thus, supposing A atomic, <u>setting</u> $[\![A]\!] = \mathcal{A}$ induces a functor $[\![-]\!]$ out of this free category which assigns (21.2) to (21.1)). Suppose now instead that $A = {!x}$ with x atomic, and choose

$$[\![x]\!] = \mathrm{mod}_k, \text{ so } [\![A]\!] = [\![!x]\!] = ![\![x]\!] = !\mathrm{mod}_k.$$

Then (21.2) is now a k-linear functor

$$[\![\underline{2}]\!] : ![\,!\mathrm{mod}_k, !\mathrm{mod}_k]_k \longrightarrow [\,!\mathrm{mod}_k, !\mathrm{mod}_k]_k. \qquad (22.2)$$

Any polynomial functor $X \in \mathcal{P}_d$, via (17.1)–(17.3), gives k-linear $\Phi(\widetilde{X}) : !\mathrm{mod}_k \to \mathrm{mod}_k$, which by the universal property gives $\Phi(\widetilde{X})_{\mathrm{lift}} : !\mathrm{mod}_k \longrightarrow !\mathrm{mod}_k$. Then we may compute

$$[\![\underline{2}]\!]\big(\Phi(\widetilde{X})_{\mathrm{lift}} \big) = \Phi(\widetilde{X})_{\mathrm{lift}} \circ \Phi(\widetilde{X})_{\mathrm{lift}} \qquad (22.3)$$

But considering

$$\begin{tikzcd} !\mathrm{mod}_k \arrow[r, "\Phi(\tilde{X})"] \arrow[dr, dashed, bend right, "\Phi(\tilde{X})_{\mathrm{lift}}"'] & \mathrm{mod}_k & \mathrm{mod}_k \\ & !\mathrm{mod}_k \arrow[u, "d"] \arrow[ur, "\Phi(\tilde{X})"] \arrow[r, dashed, "\Phi(\tilde{X})_{\mathrm{lift}}"'] & !\mathrm{mod}_k \arrow[u, "d"'] \end{tikzcd} \qquad (23.1)$$

we see that $\Phi(\tilde{X})^2_{\mathrm{lift}} = [\Phi(\tilde{X}) \circ \Phi(\tilde{X})_{\mathrm{lift}}]_{\mathrm{lift}}$, which by p. 16 is equal to the lift of $\Phi(\widetilde{X^2}) : !\mathrm{mod}_k \longrightarrow \mathrm{mod}_k$. In conclusion then,

$$[1\,2\,1]\big(\Phi(\tilde{X})_{\mathrm{lift}}\big) = \Phi(\widetilde{X^2})_{\mathrm{lift}} \qquad (23.2)$$

so that once we have properly encoded polynomial functors $\mathrm{mod}_k \longrightarrow \mathrm{mod}_k$ as endofunctors of $!\mathrm{mod}_k$ the Church numeral $\underline{2}$ (resp. $\underline{n}$) acts by squaring (resp. raising to the nth power).

<u>Example 3</u> The fully faithful functor $\Phi^d : \mathrm{rep}\,T^d_k \longrightarrow [\,!\mathrm{mod}_k, \mathrm{mod}_k]_k$ of p. 12 has the property that $\mathrm{Hom}(\mathrm{Im}\,\Phi^d, \mathrm{Im}\,\Phi^e) = 0$ for $d \neq e > 0$. A natural transformation $\Phi^d(\tilde{X}) \longrightarrow \Phi^e(\tilde{Y})$ would be the same data as a natural transformation $X \longrightarrow Y$ which is zero if $e \neq d$ (TODO check this). Hence we have a fully faithful

$$\mathcal{P} \cong \bigoplus_{d>0} \mathrm{rep}\,T^d_k \hookrightarrow [\,!\mathrm{mod}_k, \mathrm{mod}_k]_k \qquad (23.3)$$

and moreover the above embeds $\mathcal{P}$ as a <u>monoidal category</u> (under composition)

$$\mathcal{P} \xhookrightarrow{\Phi(\widetilde{-})} [\,!\mathrm{mod}_k, !\mathrm{mod}_k]_k \qquad (23.4).$$

Now we have <u>as algebras</u> (see for example [11, Proposition 3.2])

$$K_0(\mathcal{P}) \cong B \tag{24.1}$$

where B is the ring of symmetric functions in infinitely many variables over $\mathbb{Z}$. In particular for each partition λ we have the Schur function $s_\lambda \in B$ and a corresponding $S_\lambda \in \mathcal{P}$ s.t. $[S_\lambda]$ maps to s_λ, and $S_\mu \circ S_\lambda$ is computed (at least in K_0 TODO) via the Littlewood-Richardson coefficients. This gives us interesting combinatorics for composition in $[!\mathrm{mod}_k, !\mathrm{mod}_k]_k$ which can be fed into proofs like (21.1). Possibly B can be used to model linear logic in this way, entirely in terms of symmetric functions and their combinatorics.

<u>Example 4</u> From [2, §3.2] we may encode a sequence $S \in \{0,1\}^*$ as a proof $\underline{S}$ of

$$\underline{\mathrm{bint}}_A := \,!(A \multimap A) \multimap (\,!(A \multimap A) \multimap (A \multimap A)). \tag{24.2}$$

Taking $A = \,!x$ as above and $[\![x]\!] = \mathrm{mod}_k$, and using the embedding of (23.4), the semantics of $\underline{S}$ is a functor

$$[\![\underline{S}]\!] : \mathcal{P} \times \mathcal{P} \longrightarrow \mathcal{P} \tag{24.3}$$

given for example on $S = 001$ by

$$[\![\underline{001}]\!](X, Y) = Y \circ X \circ X \tag{24.4}$$

If, say, $X(V) = S^d V$ and $Y(V) = \Lambda^e(V)$ then $[\![\underline{001}]\!](S^d, \Lambda^e)$ is the functor $V \longmapsto \Lambda^e(S^d(S^d V))$.

On natural transformations $\phi : X \Rightarrow X'$, $\psi : Y \Rightarrow Y'$ we have

$$[\![1\ \underline{001}\ 1]\!](\phi, \psi) = \psi * \phi * \phi. \qquad (25.1)$$

In the concrete example $X = S^d$, $Y = \Lambda^e$, and say $X = X'$, so $\phi : S^d \Rightarrow S^d$, $\psi : \Lambda^e \Rightarrow \Lambda^e$, we have

$$\begin{tikzcd} \mathrm{mod}_k \arrow[r, bend left, "S^d"] \arrow[r, bend right, "S^d"'] \arrow[r, phantom, "\Downarrow\phi"] & \mathrm{mod}_k \arrow[r, bend left, "S^d"] \arrow[r, bend right, "S^d"'] \arrow[r, phantom, "\Downarrow\phi"] & \mathrm{mod}_k \arrow[r, bend left, "\Lambda^e"] \arrow[r, bend right, "\Lambda^e"'] \arrow[r, phantom, "\Downarrow\psi"] & \mathrm{mod}_k \end{tikzcd}$$

$$\begin{aligned} \psi * \phi * \phi &= (\psi * \phi) * \phi \\ &= [(\psi S^d) \circ (\Lambda^e \phi)] * \phi \\ &= [(\psi S^d) \circ (\Lambda^e \phi)] S^d \circ S^d S^d \phi \\ &= \psi S^d S^d \circ \Lambda^e \phi S^d \circ S^d S^d \phi \end{aligned}$$

and hence

$$[\![1\ \underline{001}\ 1]\!](\phi, \psi)_V = (\psi * \phi * \phi)_V : \Lambda^e(S^d(S^d V)) \longrightarrow \Lambda^e(S^d(S^d V))$$

is equal to

$$\psi_{S^d(S^d V)} \circ \Lambda^e(\phi_{S^d V}) \circ S^d S^d(\phi_V) \qquad (25.2)$$

This shows that

$$[\![1\ \underline{001}\ 1]\!] : \mathrm{Nat}(S^d, S^d) \times \mathrm{Nat}(\Lambda^e, \Lambda^e) \longrightarrow \mathrm{Nat}(\Lambda^e S^d S^d, \Lambda^e S^d S^d)$$

is itself polynomial in the inputs ϕ, ψ, and moreover the polynomials we obtain are related to the square of the polynomials computing S^d on Hom-spaces, multiplied with the polynomials computing Λ^e. The precise polynomials we get are detailed by the linear maps of the form (21.4), and we read off which linear maps from the shape of the proof 001. We now explain this.

In this case [2, Example 3.10] shows that $[\![1001]\!]$ is the functor (writing $\mathcal{A}$ for $!\mathrm{mod}_k$) shown below:

$$\begin{array}{ccc} ![\mathcal{A},\mathcal{A}]_k \otimes ![\mathcal{A},\mathcal{A}]_k & \xrightarrow{\Delta \otimes 1} & ![\mathcal{A},\mathcal{A}]_k \otimes ![\mathcal{A},\mathcal{A}]_k \otimes ![\mathcal{A},\mathcal{A}]_k \\ & & \Big\downarrow d\otimes d \otimes d \\ & & [\mathcal{A},\mathcal{A}]_k \otimes [\mathcal{A},\mathcal{A}]_k \otimes [\mathcal{A},\mathcal{A}]_k \\ & & \Big\downarrow \mathrm{comp}^2 \quad \circledast \\ & & [\mathcal{A},\mathcal{A}]_k \end{array} \qquad (26.1)$$

applied to $\Phi(\tilde{X})_{\mathrm{lift}} \otimes \Phi(\tilde{Y})_{\mathrm{lift}}$ this gives $\Phi(\widetilde{YXX})_{\mathrm{lift}}$. In the final step $\circledast$ the functors are encoded via the tensors making up $\tilde{X}, \tilde{Y}$, and these are contracted in an appropriate way with the coefficients of ϕ, ψ to form (25.2).

<u>Upshot</u> The semantics of linear logic proofs construct <u>new polynomial functors from old ones</u>, and in a functorial way. Moreover, the semantics also computes the polynomial functions which compute these functorial constructions on the level of natural transformations.

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