MAST90132 Assignment 3

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0.1 Question 1 (LB1-2)

Part (i): Assume that V is a normed vector space and that $v \in V$. Define the map $\eta_V : \mathbb{F} \to V$ by

$$\eta_v : \mathbb{F} \to V$$

$$\lambda \mapsto \lambda v$$

We will demonstrate that $\|\eta_v\| = \|v\|$. Using the definition of the operator norm, we will expand the LHS as follows:

$$\begin{aligned}
\|\eta_v\| &= \sup_{|\lambda|=1} \|\eta_v(\lambda)\| \\
&= \sup_{|\lambda|=1} \|\lambda v\| \\
&= \sup_{|\lambda|=1} |\lambda| \|v\| \\
&= \|v\|.
\end{aligned}$$

Hence, $\|\eta_v\| = \|v\|$.

Part (ii): Define the map Φ as follows:

$$\Phi : V \to \beta(\mathbb{F}, V)$$

$$v \mapsto \eta_v$$

Here $\beta(\mathbb{F}, V)$ is the set of all bounded linear operators from $\mathbb{F}$ to V. We will prove that Φ is an isometric (norm-preserving) isomorphism of the normed vector spaces V and $\beta(\mathbb{F}, V)$.

To show: (a) Φ is a linear map.

(b) Φ is injective.

(c) Φ is surjective.

(d) For all $v \in V$, $\|v\| = \|\Phi(v)\|$.

(a) Assume that $v, w \in V$ and that $\alpha \in \mathbb{F}$. First, observe that for all $\lambda \in \mathbb{F}$,

$$\begin{aligned}\Phi(v+w)(\lambda) &= \eta_{v+w}(\lambda) \\ &= \lambda(v+w) \\ &= \lambda v + \lambda w \\ &= \eta_v(\lambda) + \eta_w(\lambda) \\ &= \Phi(v)(\lambda) + \Phi(w)(\lambda).\end{aligned}$$

Hence, $\Phi(v+w) = \Phi(v) + \Phi(w)$. For scalar multiplication, we proceed with a similar calculation as above.

$$\begin{aligned}\Phi(\alpha v)(\lambda) &= \eta_{\alpha v}(\lambda) \\ &= \lambda(\alpha v) \\ &= \alpha(\lambda v) \\ &= \alpha\eta_v(\lambda) \\ &= \alpha\Phi(v)(\lambda).\end{aligned}$$

Therefore, $\Phi(\alpha v) = \alpha\Phi(v)$. The two calculations above demonstrate that Φ is linear.

(b) Assume that $v, w \in V$ and that $\Phi(v) = \Phi(w)$. Then, $\eta_v = \eta_w$ and so, for all $\lambda \in \mathbb{F}$,

$$\lambda v = \eta_v(\lambda) = \eta_w(\lambda) = \lambda w.$$

Since $\lambda v = \lambda w$ for all $\lambda \in \mathbb{F}$, it must hold whenever $\lambda \neq 0$. Then, $\lambda(v-w) = 0$ and consequently, $v - w = 0$. So, $v = w$ and as a result, Φ must injective.

(c) Assume that $S \in \beta(\mathbb{F}, V)$. Then, for each $\lambda \in \mathbb{F}$, S sends λ to an arbitrary vector $S(\lambda)$ in V. Since S is linear, for all $\alpha, \beta \in \mathbb{F}$, we have

$$S(\alpha\beta) = \alpha S(\beta).$$

In particular, when $\beta = 1$, then $S(\alpha) = \alpha S(1)$. Keeping this in mind, we select the vector $S(1) \in V$. Then, for all $\alpha \in \mathbb{F}$,

$$\begin{aligned}\Phi(S(1))(\alpha) &= \eta_{S(1)}(\alpha) \\ &= \alpha S(1) \\ &= S(\alpha).\end{aligned}$$

So, $\Phi(S(1)) = S$. This reveals that Φ is surjective.

(d) From part (i), we have $\|\Phi(v)\| = \|\eta_v\| = \|v\|$ for all $v \in V$.

By combining parts (a) to (d) of the proof, we deduce that Φ is an isometric (norm preserving) vector space isomorphism between V and $\beta(\mathbb{F}, V)$.

Part (iii): Assume that V and W are normed vector spaces over the field $\mathbb{F}$. As a slight adaption of notation, let $\Phi_V : V \to \beta(\mathbb{F}, V)$ be the isometric isomorphism defined in the previous part. First, we will show that Φ_V and its inverse Φ_V^{-1} are continuous.

To show: (a) Φ_V is a continuous map.

(b) Φ_V^{-1} is a continuous map.

(a) Assume that $\epsilon \in \mathbb{R}_{>0}$. Take $x, y \in V$ and $\delta = \epsilon$ so that $\|x - y\| < \epsilon$. Then,

$$\begin{aligned}\|\Phi_V(x) - \Phi_V(y)\| &= \|\Phi_V(x - y)\| \quad \text{(Linearity)} \\ &= \|x - y\| \quad \text{(Part (i))} \\ &< \epsilon.\end{aligned}$$

Hence, Φ_V is a continuous map.

(b) Assume that $\epsilon \in \mathbb{R}_{>0}$. Take $P, Q \in \beta(\mathbb{F}, V)$. From part (ii) of this question, we can write $P = \Phi_V(P(1))$ and $Q = \Phi_V(Q(1))$ by the surjectivity of Φ_V. Now set $\delta = \epsilon$ so that $\|P - Q\| = \|\Phi_V(P(1)) - \Phi_V(Q(1))\| < \epsilon$. Then,

$$\begin{aligned}
\|\Phi_V^{-1}(P) - \Phi_V^{-1}(Q)\| &= \|\Phi_V^{-1}(\Phi_V(P(1))) - \Phi_V^{-1}(\Phi_V(Q(1)))\| \\
&= \|P(1) - Q(1)\| \\
&= \|(P - Q)(1)\| \\
&\leq \sup_{|\lambda|=1} \|(P - Q)(\lambda)\| \\
&= \|P - Q\| \\
&< \epsilon.
\end{aligned}$$

Therefore, Φ_V^{-1} is also a continuous map.

Let $T \in \beta(V, W)$. Define the map $\Omega : \beta(V, W) \times V \to W$ by

$$\Omega(T, v) = T(v).$$

We can write Ω as the following composite:

$$\beta(V, W) \times V \xrightarrow{id \times \Phi_V} \beta(V, W) \times \beta(\mathbb{F}, V) \xrightarrow{\circ} \beta(\mathbb{F}, W) \xrightarrow{\Phi_W^{-1}} W$$

In the above composite, $\circ$ denotes the composition of two bounded linear operators:

$$\circ : \beta(V, W) \times \beta(\mathbb{F}, V) \to \beta(\mathbb{F}, W)$$

$$S \times U \mapsto S \circ U$$

The map $id : \beta(V, W) \to \beta(V, W)$ denotes the identity map on $\beta(V, W)$. To see that the above composite agrees with Ω, we apply each step of the composite to $(T, v) \in \beta(V, W) \times V$:

$$(T, v) \mapsto (T, \eta_v) \mapsto T \circ \eta_v = \eta_{T(v)} \mapsto T(v).$$

The last step of the composite requires justification. For all $\lambda \in \mathbb{F}$,

$$\begin{aligned}
(T \circ \eta_v)(\lambda) &= T(\eta_v(\lambda)) \\
&= T(\lambda v) \\
&= \lambda T(v) \\
&= \eta_{T(v)}(\lambda).
\end{aligned}$$

So, $T \circ \eta_v = \eta_{T(v)}$ and thus, the composite agrees with Ω. To see that Ω is continuous, note that it is the composite of continuous maps.

1. Φ_V is continuous from part (a) of this particular question. The identity map id is also continuous by a similar argument to part (a). Since the product of continuous functions is continuous, $id \times \Phi_V$ is also continuous.

2. The composition map $\circ$ is continuous as proven in lectures.

3. Φ_W^{-1} is also a continuous map by applying part (b) of this question.

Since Ω is the composite of continuous maps, Ω must therefore be continuous.

Part (iv): Assume that $T : V \to V$ is a bounded linear operator, where V is a Banach space. We want to show that for all $v \in V$,

$$\exp(T)(v) = \lim_{n\to\infty} \sum_{i=0}^{n} \frac{T^i(v)}{i!}.$$

From the lectures, we know that the sequence of partial sums converges absolutely in $\beta(V, V)$ and thus, converges:

$$s_m = \sum_{i=0}^{m} \frac{T^i}{i!}.$$

Hence, we can write

$$\exp(T) = \sum_{i=0}^{\infty} \frac{T^i}{i!} = \lim_{m\to\infty} s_m.$$

We also know from part (iii) that the map $\Omega : \beta(V, V) \times V \to V$ is also continuous. Hence, it commutes with limits. Hence, we can express $\exp(T)(v)$ as

$$
\begin{aligned}
\exp(T)(v) &= \Omega(\exp(T), v) \\
&= \Omega(\sum_{i=0}^{\infty} \frac{T^i}{i!}, v) \\
&= \Omega(\lim_{m\to\infty} \sum_{i=0}^{m} \frac{T^i}{i!}, v) \\
&= \lim_{m\to\infty} \Omega(\sum_{i=0}^{m} \frac{T^i}{i!}, v) \quad \text{(Continuity of } \Omega\text{)} \\
&= \lim_{m\to\infty} \sum_{i=0}^{m} \frac{T^i(v)}{i!}.
\end{aligned}
$$

0.2 Question 2 (LB1-5)

Assume that $A \in M_{n\times n}(\mathbb{C})$. By using the Jordan normal form of A, we will demonstrate that $\det(\exp(A)) = \exp(Tr(A))$.

The eigenvalues of A, which we will denote by λ_i for all $i \in \{1, \ldots, n\}$, must exist in the field $\mathbb{C}$, since $\mathbb{C}$ is algebraically closed. So, there exists $P \in GL_n(\mathbb{C})$ such that $A = PJP^{-1}$, where J is the Jordan normal form of A (an upper triangular matrix with the eigenvalues of A along its diagonal). Observe that by taking the trace of both sides, we find that

$$Tr(A) = Tr(PJP^{-1}) = Tr(JP^{-1}P) = Tr(J) = \lambda_1 + \cdots + \lambda_n$$

since $Tr(XY) = Tr(YX)$ for all $X, Y \in M_{n\times n}(\mathbb{C})$. Now, apply the exponential map to A, which yields

$$\begin{aligned}\exp(A) &= \exp(PJP^{-1}) \\ &= \sum_{i=0}^{\infty} \frac{(PJP^{-1})^i}{i!} \\ &= \sum_{i=0}^{\infty} \frac{PJ^iP^{-1}}{i!} \\ &= P(\sum_{i=0}^{\infty} \frac{J^i}{i!})P^{-1} \\ &= Pe^JP^{-1}.\end{aligned}$$

To simplify this, we can further decompose J as the sum $D + N$, where D is the diagonal matrix $diag[\lambda_1, \ldots, \lambda_n]$ and N is a nilpotent, upper triangular matrix with zeros across its diagonal. By applying the exponential map once again, we find that $e^D = diag[e^{\lambda_1}, \ldots, e^{\lambda_n}]$ and that e^N is an upper triangular matrix with ones along its diagonal (this reveals that $\det(e^N) = 1$).

Since $DN = ND$ (as D is a diagonal matrix), $\exp(D + N) = \exp(D)\exp(N)$ and consequently, $\exp(A) = Pe^{D+N}P^{-1} = Pe^De^NP^{-1}$. We can now take the determinant of both sides to get

$$
\begin{aligned}
\det(\exp(A)) &= \det(P)\det(e^{D})\det(e^{N})[\det(P)]^{-1} \\
&= \det(P)\det(e^{D})(1)[\det(P)]^{-1} \quad \text{(Definition of } e^{N}\text{)} \\
&= \det(e^{D}) \\
&= e^{\lambda_1+\cdots+\lambda_n} \\
&= \exp(Tr(A)).
\end{aligned}
$$

0.3 Question 3 (LB1-6)

Our first observation in this question is that the matrices X, Y and H are all nilpotent. In fact, a quick calculation reveals that $X^2 = Y^2 = H^2 = 0$. Assume now that $\alpha \in \mathbb{R}$. Then, we can compute

$$\begin{aligned}\exp(\alpha X) &= I + \alpha X + \frac{(\alpha X)^2}{2!} + O(X^3) \\ &= I + \alpha X \\ &= \begin{pmatrix} 1 & \alpha & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},\end{aligned}$$

$$\begin{aligned}\exp(\alpha Y) &= I + \alpha Y + \frac{(\alpha Y)^2}{2!} + O(X^3) \\ &= I + \alpha Y \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \alpha \\ 0 & 0 & 1 \end{pmatrix}\end{aligned}$$

and

$$\begin{aligned}\exp(\alpha H) &= I + \alpha H + \frac{(\alpha H)^2}{2!} + O(X^3) \\ &= I + \alpha H \\ &= \begin{pmatrix} 1 & 0 & \alpha \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.\end{aligned}$$

Note: The use of Big O notation in the above calculations is to collect all the higher order terms in the infinite series expansions. This is **not** the same as how Big O notation was used to prove the Lie product formula in lectures.

0.4 Question 4 (L5-3)

Assume that $\mathcal{H}$ is a finite dimensional inner product space and $T : \mathcal{H} \to \mathcal{H}$ be a linear operator. We will demonstrate that T is skew self-adjoint if and only if for all $\alpha \in \mathbb{R}$, $e^{\alpha T}$ is unitary.

To show: (a) If T is skew self-adjoint, then $e^{\alpha T}$ is unitary for all $\alpha \in \mathbb{R}$.

(b) If $e^{\alpha T}$ is unitary for all $\alpha \in \mathbb{R}$, then T is skew self-adjoint.

(a) Assume that T is skew self-adjoint and $\alpha \in \mathbb{R}$. Assume that $x, y \in \mathcal{H}$. We want to show that $\langle e^{\alpha T}x, e^{\alpha T}y\rangle = \langle x, y\rangle$. We compute the LHS directly as

$$\begin{aligned}
\langle e^{\alpha T}x, e^{\alpha T}y\rangle &= \langle \lim_{m\to\infty} \sum_{i=0}^{m} \frac{\alpha^i T^i x}{i!}, e^{\alpha T}y\rangle \\
&= \lim_{m\to\infty} \langle \sum_{i=0}^{m} \frac{\alpha^i T^i x}{i!}, e^{\alpha T}y\rangle \quad (\langle -, v\rangle \text{ is continuous}) \\
&= \lim_{m\to\infty} \sum_{i=0}^{m} \frac{\alpha^i}{i!}\langle T^i x, e^{\alpha T}y\rangle.
\end{aligned}$$

We would like to apply skew self-adjointness of T to the above line. The way we will do this is encapsulated below:

To show: (aa) For all $i \in \mathbb{Z}_{>0}$, $\langle T^i x, y\rangle = (-1)^i\langle x, T^i y\rangle$.

(aa) We can prove this by induction. For the base case, assume that $i = 1$. Due to the assumption that T is skew self-adjoint, we have

$$\langle Tx, y\rangle = -\langle x, Ty\rangle = (-1)^1\langle x, Ty\rangle$$

as required. This proves the base case.

For the inductive hypothesis, assume that for some $k \in \mathbb{Z}_{>0}$, $\langle T^k x, y\rangle = (-1)^k\langle x, T^k y\rangle$. Then, observe that

$$\langle T^{k+1}x, y\rangle = \langle T(T^k x), y\rangle = -\langle T^k x, Ty\rangle = (-1)^{k+1}\langle x, T^{k+1}y\rangle.$$

This completes the induction.

(a) Using part (aa), we can proceed with our calculation as follows:

$$\begin{aligned}
\lim_{m\to\infty}\sum_{i=0}^{m}\frac{\alpha^i}{i!}\langle T^i x, e^{\alpha T}y\rangle &= \lim_{m\to\infty}\sum_{i=0}^{m}\frac{\alpha^i(-1)^i}{i!}\langle x, T^i e^{\alpha T}y\rangle \\
&= \lim_{m\to\infty}\langle x, \sum_{i=0}^{m}\frac{\alpha^i(-1)^i}{i!}T^i e^{\alpha T}y\rangle \\
&= \langle x, \lim_{m\to\infty}\sum_{i=0}^{m}\frac{(-\alpha T)^i}{i!}e^{\alpha T}y\rangle \quad (\langle v, -\rangle \text{ is continuous}) \\
&= \langle x, e^{-\alpha T}e^{\alpha T}y\rangle \\
&= \langle x, y\rangle.
\end{aligned}$$

Hence, for all $x, y \in \mathcal{H}$ and $\alpha \in \mathbb{R}$, $\langle e^{\alpha T}x, e^{\alpha T}y\rangle = \langle x, y\rangle$. So, $e^{\alpha T}$ must be a unitary operator on $\mathcal{H}$ for all $\alpha \in \mathbb{R}$.

(b) For the converse, assume that for all $\alpha \in \mathbb{R}$, $e^{\alpha T}$ is a unitary operator on $\mathcal{H}$. This means that for all $x, y \in \mathcal{H}$,

$$\langle e^{\alpha T}x, y\rangle = \langle x, e^{-\alpha T}y\rangle.$$

Roughly speaking, the adjoint operator of $e^{\alpha T}$ is the inverse $e^{-\alpha T}$. Differentiating both sides of the equation with respect to α yields (in tandem with the continuity of the inner product),

$$\begin{aligned}
\frac{d}{d\alpha}(\langle e^{\alpha T}x, y\rangle) &= \frac{d}{d\alpha}(\langle x, e^{-\alpha T}y\rangle) \\
\langle \frac{d}{d\alpha}(e^{\alpha T}x), y\rangle &= \langle x, \frac{d}{d\alpha}(e^{-\alpha T}y)\rangle \quad \text{(Continuity of inner product)} \\
\langle Te^{\alpha T}x, y\rangle &= \langle x, -Te^{-\alpha T}y\rangle \\
\langle Te^{\alpha T}x, y\rangle &= -\langle x, Te^{-\alpha T}y\rangle.
\end{aligned}$$

Setting $\alpha = 0$, we obtain $\langle Tx, y\rangle = -\langle x, Ty\rangle$. This reveals that T is skew self-adjoint as required.