MAST90132 Assignment 2

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0.1 Question 1 (L3-6)

Part (i): We state and prove the lemma below:

Lemma 0.1.1. *Define the function* $\Phi : S^2 \times [0, 2\pi) \to SO(3)$*, which sends* $(\widehat{n}, \alpha)$ *to the matrix* $R_\alpha^{\widehat{n}}$*. Then,* Φ *is surjective and continuous.*

Proof. Suppose that Φ is defined above. The goal is to show that every element of $SO(3)$ can be expressed as a rotation $R_\alpha^{\widehat{n}}$ for some $\alpha \in \mathbb{R}$ and $\widehat{n} \in S^2$.

To show: (a) For all $A \in SO(3)$, $A = R_\alpha^{\widehat{n}}$ for some $\alpha \in \mathbb{R}$ and $\widehat{n} \in \mathbb{R}^3$.

(a) Assume that $A \in SO(3)$. We claim that 1 is an eigenvalue of A.

To show: (aa) 1 is an eigenvalue of A.

(aa) Let $\lambda_1 \in \mathbb{C}$ be an eigenvalue of A. Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be the function whose associated matrix is A. Since $A \in SO(3)$, f must preserve the complex inner product. Let $u \in \mathbb{R}^3$ be an eigenvector of A, with eigenvalue λ_1. Then,

$$\langle u, u \rangle = \langle f(u), f(u) \rangle = \langle \lambda_1 u, \lambda_1 u \rangle = |\lambda_1|^2 \langle u, u \rangle.$$

So, $|\lambda_1| = 1$. Let λ_2 and λ_3 denote the other two eigenvalues of A. By the above reasoning, they must also have magnitude 1. We now have the following two cases:

Case 1: All the eigenvalues are real.

Recall that $1 = \det(A) = \lambda_1 \lambda_2 \lambda_3$. If $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$, then either all of the eigenvalues are 1 or exactly two of them are -1, which renders the last eigenvalue 1. Thus, 1 is an eigenvalue of A in this case.

Case 2: At least one of the eigenvalues are complex.

Since A is a 3×3 matrix with elements in $\mathbb{R}$, its characteristic polynomial $c_A(X)$ must be a degree 3 polynomial with real coefficients. Suppose without loss of generality that $\lambda_1 \in \mathbb{C}$ with non-zero imaginary part. Since the eigenvalues of A are roots of the real polynomial $c_A(X)$, we deduce that another eigenvalue λ_2 of A must satisfy $\lambda_2 = \overline{\lambda_1}$. Hence,

$$1 = \det(A) = \lambda_1 \overline{\lambda_1} \lambda_3 = |\lambda_1|^2 \lambda_3 = \lambda_3.$$

Once again, $1 = \lambda_3$ is an eigenvalue of A. Since we have exhausted all possible cases, we can finally deduce that 1 is an eigenvalue of $A \in SO(3)$.

(a) Since 1 is an eigenvalue, there exists a non-zero vector $v_1 \in \mathbb{R}^3$ such that $Av_1 = v_1$. By dividing both sides by the norm of v_1, we can assume without loss of generality that v_1 is a unit vector in the unit sphere S^2. Take two other vectors $v_2, v_3 \in \mathbb{R}^3$. We can use the Gram-Schmidt procedure to obtain an orthonormal basis $\{v_1, \tilde{v_2}, \tilde{v_3}\}$ of $\mathbb{R}^3$. Now consider the 3×3 matrix $B = [v_1, \tilde{v_2}, \tilde{v_3}]$. Since the columns of B form an orthonormal basis, B must be an orthogonal matrix. By permuting the second and third columns, we can assume that $\det(B) = 1$ and thus, that $B \in SO(3)$.

Now, we compute the matrix A with respect to our new orthonormal basis $\{v_1, \tilde{v_2}, \tilde{v_3}\}$. First consider the vector subspace $span(v_1)$. It orthogonal complement is given by

$$(span(v_1))^{\perp} = \{v \in \mathbb{R}^3 \mid v \cdot v_1 = 0\} = span(\tilde{v_2}, \tilde{v_3}).$$

Our second observation is that since $SO(3)$ is a group, the matrix product $AB = [Av_1, A\tilde{v_2}, A\tilde{v_3}] = [v_1, A\tilde{v_2}, A\tilde{v_3}] \in SO(3)$. As a result of this, the vectors $A\tilde{v_2}, A\tilde{v_3} \in (span(v_1))^{\perp} = span(\tilde{v_2}, \tilde{v_3})$, since the columns of AB form an orthonormal basis of $\mathbb{R}^3$. Hence, we can write $A\tilde{v_2} = a\tilde{v_2} + b\tilde{v_3}$ and $A\tilde{v_3} = c\tilde{v_2} + d\tilde{v_3}$ for some $a, b, c, d \in \mathbb{R}$.

Using this information, we can now write the matrix A with respect to our constructed orthonormal basis $\{v_1, \tilde{v_2}, \tilde{v_3}\}$. We find that

$$B^{-1}AB = \begin{pmatrix} 1 & & \\ & a & b \\ & c & d \end{pmatrix}.$$

Since $A, B \in SO(3)$, the above matrix is also in $SO(3)$, revealing that the submatrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \in SO(2)$$

for some $\theta \in \mathbb{R}$. So, $B^{-1}AB = R_\theta^x$. Therefore, $A = BR_\theta^x B^{-1}$. Due to the change of basis induced by the matrix $B \in SO(3)$ (notably, it sends the x-axis to v_1), we can interpret A as a rotation by θ about an axis in the direction of v_1. Therefore, $A = R_\theta^{v_1}$. Hence, every matrix in $SO(3)$ is a rotation matrix. In particular, every matrix in $SO(3)$ is a rotation matrix whose axis of rotation is the eigenvector with eigenvalue 1.

(a) Now assume that $C \in SO(3)$. Then, from part (aa), there exists $\theta \in [0, 2\pi)$ and $\widehat{w} \in S^2$ such that $C = R_\theta^{\widehat{w}}$. Consequently, $\Phi(\widehat{w}, \theta) = R_\theta^{\widehat{w}} = C$, revealing that Φ is surjective. To see that it is continuous, we first identify the set of 3×3 matrices $M_{3\times 3}(\mathbb{R})$ with $\mathbb{R}^9$ (this is by "reading off" the elements of a matrix). Take $\Phi(\widehat{z}, \eta) = R_\eta^{\widehat{z}}$. Note that the entries of the matrix $R_\eta^{\widehat{z}}$ are linear combinations of trigonometric functions, which are continuous. Since all of the component functions of Φ are continuous in this case, we deduce that Φ must also be a continuous function.

$\square$

Part (ii): Assume that $R_\alpha^{\widehat{n}}$ and $R_\beta^{\widehat{m}}$ are rotation matrices in $SO(3)$. Assume that $\sim$ was defined as above. In order to give an explicit description of $\sim$ on the set $S^2 \times [0, 2\pi)$, it suffices to know when $R_\alpha^{\widehat{n}} = R_\beta^{\widehat{m}}$. The claim here is that

To show: (a) If $R_\alpha^{\widehat{n}} = R_\beta^{\widehat{m}}$, where $\alpha, \beta \in [0, 2\pi)$ and $\widehat{n}, \widehat{m} \in S^2$, then the following three cases are possible:

1. $\alpha = \beta = 0$.
2. $\widehat{m} = -\widehat{n}$ and $\beta = -\alpha$
3. $\alpha = \beta$ and $\widehat{m} = \widehat{n}$

(a) Assume that $R_\alpha^{\widehat{n}} = R_\beta^{\widehat{m}}$, with $\alpha, \beta \in [0, 2\pi)$ and $\widehat{n}, \widehat{m} \in S^2$. Then, $I = {R_\alpha^{\widehat{n}}}^{-1} R_\beta^{\widehat{m}}$, where I is the 3×3 identity matrix. Recall from the previous result that $R_\beta^{\widehat{m}}$ has an eigenvalue of 1, with corresponding eigenvector $\widehat{m}$. By applying this, we obtain

$$\widehat{m} = I\widehat{m} = {R_\alpha^{\widehat{n}}}^{-1} R_\beta^{\widehat{m}} \widehat{m} = {R_\alpha^{\widehat{n}}}^{-1} \widehat{m}.$$

Thus, $\widehat{m}$ is also an eigenvector of ${R_\alpha^{\widehat{n}}}^{-1}$ with eigenvalue 1. But, ${R_\alpha^{\widehat{n}}}^{-1} = R_{-\alpha}^{\widehat{n}}$. Again, the previous result shows that for all $\alpha \in [0, 2\pi)$, $\widehat{n}$ is an eigenvector with eigenvalue 1, just like $\widehat{m}$.

Observe that for all rotation matrices $R_\alpha^{\widehat{n}} \in SO(3)$, if 1 is an eigenvalue with multiplicity greater than 1, then all of the eigenvalues of $R_\alpha^{\widehat{n}}$ are 1 because $\det(R^{\widehat{n}}_\alpha) = 1$. In this case, $R^{\widehat{n}}_\alpha$ must be the identity matrix I. So, $\alpha = 0$ for this to be true. Translating this back to the original scenario, we deduce that $\alpha = \beta = 0$.

For the other case, where the eigenvalue of 1 occurs with multiplicity 1 in the matrix $R^{\widehat{n}}_{-\alpha}$, then $\widehat{n}$ is the rotation axis for $R^{\widehat{n}}_{-\alpha}$. But, $\widehat{m}$ is also an eigenvector with eigenvalue 1. So, $\widehat{m}$ is also an axis of rotation for $R^{\widehat{n}}_{-\alpha}$. This establishes that either $\widehat{m} = \widehat{n}$ or $\widehat{m} = -\widehat{n}$. In the first case, we must have $\alpha = \beta$ for $R^{\widehat{n}}_\alpha = R^{\widehat{m}}_\beta$ to be true. For the second case, we similarly must have $\alpha = -\beta$.

Now we can explicity describe the relation $\sim$ on $S^2 \times [0, 2\pi)$. We say that $(\widehat{n}, \alpha) \sim (\widehat{m}, \beta)$ if any one of the three conditions below are satisfied:

1. $\alpha = \beta = 0$.
2. $\widehat{m} = \widehat{n}$ and $\alpha = \beta$.
3. $\widehat{m} = -\widehat{n}$ and $\alpha = -\beta$.

0.2 Question 2 (L3B-2)

Fundamentally, this question is asking about a specific case of a *ringed space*. The following results will make this remark precise.

Lemma 0.2.1 (Restriction of a smooth map). *Let M be a smooth n-manifold and W be an open subset of M, with respect to the manifold topology on M. Let $f : W \to \mathbb{R}$ be a smooth function. If W' is another open subset of M such that $W' \subseteq W$, then the restriction $f|_{W'} : W' \to \mathbb{R}$ is also a smooth function.*

Proof. Assume that M is a smooth n-manifold with atlas $\{(U_a, \varphi_a)\}_{a \in I}$ and that W is an open subset of M, with respect to the manifold topology. Assume that $f : W \to \mathbb{R}$ is smooth. Then, for all $w \in W$, there exists a coordinate chart (U_b, φ_b) such that $w \in U_b$ and $f \circ \varphi_b^{-1}$ is smooth as a function from $\varphi_b(U_b) \subseteq \mathbb{R}^n$ to $\mathbb{R}$. We observe that its restriction $f|_{W'} : W' \to \mathbb{R}$ to the open set $W' \subseteq W$ can be described by the following composite:

$$W' \hookrightarrow W \xrightarrow{f} \mathbb{R}$$

Here, the first arrow denotes the inclusion map $\iota : W' \to W$. Since the composition of smooth functions is smooth, it suffices to show that ι is a smooth function. Assume that $w' \in W'$. Since $W' \subseteq W$, there exists a coordinate chart (U_b, φ_b) such that $w' \in U_b$ and the composite

$$\varphi_b(W' \cap U_b) \xrightarrow{\varphi_b^{-1}} W' \hookrightarrow W \xrightarrow{\varphi_b} \varphi_b(W \cap U_b)$$

is the inclusion of the open set $\varphi_b(W' \cap U_b)$ in $\varphi_b(W \cap U_b)$. This is smooth because each of the component functions of this composite is smooth (each component function is the polynomial $j(x) = x$). Hence, the inclusion map $\iota : W' \to W$ is smooth and consequently, $f|_{W'}$ must also be smooth because it is the composition of smooth functions.

□

Theorem 0.2.2 (Presheaf on a Smooth Manifold). *Let (X, τ) be a topological space, where X is a smooth n-manifold and τ is the manifold topology on X. Let $\mathscr{C}$ denote the category with open subsets of X as the objects and inclusion maps i_U^V as the morphisms. The inclusion maps are defined as follows: If $U \subseteq V$, then*

$$i_U^V(U) = V.$$

*Then, there exists a contravariant functor $\mathcal{O}_X : \mathscr{C} \to$ **Rng** (**Rng** is the category of rings) such that each open set U is mapped to the commutative ring of real-valued smooth functions on U and each inclusion map is mapped to restriction maps. More succinctly, for all open subsets $U, V \subseteq X$ such that $U \subseteq V$,*

$$\mathcal{O}_X(U) = \{f : U \to \mathbb{R} \mid f \text{ is smooth}\} \text{ and } \mathcal{O}_X(i_U^V) = res_U^V$$

In turn, each restriction map $res_U^V : \mathcal{O}_X(V) \to \mathcal{O}_X(U)$ is defined as

$$res_U^V(f) = f|_U.$$

Proof. Assume that X is a smooth n-manifold, τ is the manifold topology and (X, τ) is a topological space. Assume that $\mathscr{C}$ is the category of open sets in X as defined above. Assume that $\mathcal{O}_X$ is the map between $\mathscr{C}$ and **Rng** as described above.

To show: (a) If $U \in ob(\mathscr{C})$, then $\mathcal{O}_X(U) \in ob(\mathbf{Rng})$.

(b) $\mathcal{O}_X(i_U^U) = res_U^U$ and res_U^U is the identity morphism on $\mathcal{O}_X(U)$.

(c) If $U, V, W \in ob(\mathscr{C})$ with $U \subseteq V \subseteq W$, then, $res_U^W = res_U^V \circ res_V^W$.

(a) This follows from our careful definition of $\mathcal{O}_X$, since we map U to the commutative ring of smooth real-valued functions on U (recall that sums and products of smooth functions are indeed smooth).

(b) Assume that $U \in ob(\mathscr{C})$. Then, the inclusion map $i_U^U : U \to U$ is the identity morphism on U. From the definition of $\mathcal{O}_X$, $\mathcal{O}_X(i_U^U) = res_U^U$. It remains to show that res_U^U is the identity morphism defined on $\mathcal{O}_X(U)$. But, for all $f \in \mathcal{O}_X(U)$, $res_U^U(f) = f|_U = f$. Hence, res_U^U is the identity morphism of $\mathcal{O}_X(U)$.

(c) Assume that $U, V, W \in ob(\mathscr{C})$ with $U \subseteq V \subseteq W$. Assume that $f \in \mathcal{O}_X(W)$. Then, $res_U^W(f) = f|_U$. Furthermore, we observe that

$$(res_U^V \circ res_V^W)(f) = res_U^V(res_V^W(f)) = res_U^V(f|_V) = f|_U = res_U^W(f).$$

Therefore, $res_U^W = res_U^V \circ res_V^W$. We note that in the category $\mathscr{C}$, $i_U^W = i_V^W \circ i_U^V$, which means that under $\mathcal{O}_X$, the order of composition of morphisms is reversed.

Therefore, $\mathcal{O}_X$ is a contravariant functor from $\mathscr{C}$ to **Rng**. This is called a **presheaf** on the topological space (X, τ). $\square$

Theorem 0.2.3 (Ringed Space on a Smooth Manifold)**.** *Let X be a smooth n-manifold, τ be the manifold topology on X and (X, τ) be the resultant topological space. Let $\mathcal{O}_X$ be the presheaf on X which was defined in 0.2.2. Then, $\mathcal{O}_X$ is a sheaf on X and $(X, \tau, \mathcal{O}_X)$ is a ringed space.*

Proof. Assume that X is a smooth n-manifold, τ is the manifold topology on X and (X, τ) is the resultant topological space. Assume that $\mathcal{O}_X$ is the presheaf defined in 0.2.2.

To show: (a) (Locality) If $U \in \tau$, $U = \bigcup_{i \in I} V_i$ is an open cover for U and if $f_1, f_2 \in \mathcal{O}_X(U)$ such that $res^U_{V_i}(f_1) = res^U_{V_i}(f_2)$ for all $i \in I$, then $f_1 = f_2$.

(b) (Gluing) If $U \in \tau$, $U = \bigcup_{i \in I} V_i$ is an open cover for U and for all V_i, there exists $f_i \in \mathcal{O}_X(V_i)$ such that $res^{V_i}_{V_i \cap V_j}(f_i) = res^{V_j}_{V_i \cap V_j}(f_j)$ for all $i, j \in I$, then there exists a $f \in \mathcal{O}_X(U)$ such that for all $i \in I$, $res^U_{V_i}(f) = f_i$.

(a) Assume that $U \in \tau$ and $\bigcup_{i \in I} V_i$ is an open cover for U. Assume that $f_1, f_2 \in \mathcal{O}_X(U)$ such that $res^U_{V_i}(f_1) = res^U_{V_i}(f_2)$ for all $i \in I$. Since $f_1|_{V_i} = f_2|_{V_i}$ for all $i \in I$, f_1 and f_2 must agree on $\bigcup_{i \in I} V_i$. Since $U \subseteq \bigcup_{i \in I} V_i$, it must follow that f_1 and f_2 agree on U. So, $f_1 = f_2$.

(b) Assume that $U \in \tau$ and $\bigcup_{i \in I} V_i$ is an open cover for U. Assume that for all $i \in I$, there exists $f_i \in \mathcal{O}_X(V_i)$ such that $res^{V_i}_{V_i \cap V_j}(f_i) = res^{V_j}_{V_i \cap V_j}(f_j)$ for all $i, j \in I$. Assume that $x \in U$. Then, there exists a $i \in I$ such that $x \in V_i$. So, we define the function f such that $f(x) = f_i(x)$ for all $x \in V_i$ and $i \in I$.

To show: (ba) f is well defined.

(bb) For all $i \in I$, $res^U_{V_i}(f) = f_i$.

(bc) f is a smooth function.

(ba) Assume that $x \in V_i \cap V_j$ for some $i, j \in I$. Then, $f_i(x) = f_j(x)$ because $res^{V_i}_{V_i \cap V_j}(f_i) = res^{V_j}_{V_i \cap V_j}(f_j)$. So, f must be a well defined function.

(bb) From our construction of f, $res^U_{V_i}(f) = f_i$.

(bc) Since the family of functions $f_i : V_i \to \mathbb{R}$ are all smooth, we can take coordinate charts $(U_{j,i}, \varphi_{j,i})$ such that $V_i \subseteq \bigcup_j U_{j,i}$ and the map $f_i \circ \varphi^{-1}_{j,i}$ is a smooth function from $\varphi(V_i \cap U_{j,i}) \subseteq \mathbb{R}^n$ to $\mathbb{R}$. To see that f is smooth, assume that $x \in X$. Then, $x \in V_i$ for some $i \in I$. By construction, $f(x) = f_i(x)$ and so, $f \circ \varphi^{-1}_{j,i} = f_i \circ \varphi^{-1}_{j,i}$ for some coordinate chart $(U_{j,i}, \varphi_{j,i})$ containing x. Since the RHS is a smooth function from $\mathbb{R}^n$ to $\mathbb{R}$, the LHS must also be smooth. Hence, f itself is a real-valued smooth function.

Observe that f is unique. To see why, assume that there exists $g \in \mathcal{O}_X(U)$ such that for all $i \in I$, $res^U_{V_i}(g) = f_i$. Since $res^U_{V_i}(f) = f_i$ for all $i \in I$ as well, we can use the locality axiom proved in part (a) to deduce that $f = g$. Hence, f must be unique.

Therefore, $\mathcal{O}_X$ defines a sheaf on X and as a result, $(X, \tau, \mathcal{O}_X)$ is a ringed space. $\square$

Since S^2 is a smooth 2-manifold, we can apply 0.2.3 to deduce that $(S^2, \tau_{S^2}, \mathcal{O}_{S^2})$ is a ringed space, where τ_{S^2} is the manifold topology on S^2 and $\mathcal{O}_{S^2}$ is the presheaf on S^2, as depicted in 0.2.2 and 0.2.3.

Part (a): Assume that $W \subseteq S^2$ is open and that $f : W \to \mathbb{R}$ is smooth. Assume that $W' \subseteq W$ is also an open set. Applying 0.2.1, we find that the restriction $f|_{W'} : W' \to \mathbb{R}$ is also a smooth function. That is, $f|_{W'} \in C^\infty(W')$.

Part (b): Since $(S^2, \tau_{S^2}, \mathcal{O}_{S^2})$ is a ringed space by 0.2.3, it must automatically satisfy the gluing axiom.

0.3 Question 3 (L3-13)

Lemma 0.3.1 (Laplacian on S^2). *Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a smooth function. Then, on $\mathbb{R}^3 - \{0\}$,*

$$\Delta f = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial f}{\partial r}) + \frac{1}{r^2}\Delta_{S^2} f.$$

Proof. Assume that $f : \mathbb{R}^3 \to \mathbb{R}$ is a smooth function. Then, the usual Laplacian on f is given by

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.$$

We will make the substitutions to spherical coordinates:

1. $x = r\sin\theta\cos\phi$
2. $y = r\sin\theta\sin\phi$
3. $z = r\cos\theta$

where $\theta \in [0, \pi]$, $\phi \in [0, 2\pi]$ and $r > 0$. Operating within spherical coordinates, we will derive an expression for the Laplacian. First, we need to understand how to work in spherical coordinates.

Let the vector $v = (x, y, z) = (r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)$ in $\mathbb{R}^3$. Denote the three unit vectors in spherical coordinates by e_r, e_θ and e_ϕ. They are defined by the equations

$$\frac{\partial v}{\partial r} = h_r e_r, \frac{\partial v}{\partial \theta} = h_\theta e_\theta \text{ and } \frac{\partial v}{\partial \phi} = h_\phi e_\phi.$$

where h_r, h_θ and h_ϕ are scale factors. By differentiating the position vector v, we find that

$$h_r = 1, e_r = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

$$h_\theta = r, e_\theta = (\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta)$$

and

$$h_\phi = r\sin\theta, e_\phi = (-\sin\phi, \cos\phi, 0).$$

By using the usual dot product on $\mathbb{R}^3$, we find that

$$e_r \cdot e_\theta = e_r \cdot e_\phi = e_\theta \cdot e_\phi = 0.$$

Thus, spherical coordinates is an orthogonal coordinate system in $\mathbb{R}^3$. Now, we will derive the gradient ∇ and the divergence operators in spherical coordinates.

Suppose that $f : S^2(r) \to \mathbb{R}$ is a smooth function in spherical coordinates, where $S^2(r)$ denotes the sphere centred at the origin of radius $r \in (0, \infty)$. Let $\nabla f = f_r e_r + f_\theta e_\theta + f_\phi e_\phi$. We will find the component functions f_r, f_θ and f_ϕ. Our starting point is the chain rule. In spherical coordinates,

$$df = \frac{\partial f}{\partial r} dr + \frac{\partial f}{\partial \theta} d\theta + \frac{\partial f}{\partial \phi} d\phi.$$

But,

$$\begin{aligned} df &= \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz \\ &= \nabla f \cdot (dx, dy, dz) \\ &= (f_r e_r + f_\theta e_\theta + f_\phi e_\phi) \cdot dv \\ &= (f_r e_r + f_\theta e_\theta + f_\phi e_\phi) \cdot (\frac{\partial v}{\partial r} dr + \frac{\partial v}{\partial \theta} d\theta + \frac{\partial v}{\partial \phi} d\phi) \\ &= (f_r e_r + f_\theta e_\theta + f_\phi e_\phi) \cdot (h_r e_r dr + h_\theta e_\theta d\theta + h_\phi e_\phi d\phi) \\ &= h_r f_r dr + h_\theta f_\theta d\theta + h_\phi f_\phi d\phi. \end{aligned}$$

By comparing coefficients, we deduce the following set of equations:

$$f_r = \frac{1}{h_r} \frac{\partial f}{\partial r}, f_\theta = \frac{1}{h_\theta} \frac{\partial f}{\partial \theta} \text{ and } f_\phi = \frac{1}{h_\phi} \frac{\partial f}{\partial \phi}.$$

Therefore, the gradient operator in spherical coordinates is

$$\nabla f = \frac{\partial f}{\partial r} e_r + \frac{1}{r} \frac{\partial f}{\partial \theta} e_\theta + \frac{1}{r \sin\theta} \frac{\partial f}{\partial \phi} e_\phi.$$

Next, we need the divergence. Let $A = A_1 e_r + A_2 e_\theta + A_3 e_\phi$ denote a smooth vector field. We require a few properties before we proceed with the calculation. The first of these is the cross products

$$e_r \times e_\theta = e_\phi, e_r \times e_\phi = -e_\theta \text{ and } e_\theta \times e_\phi = e_r.$$

These can be verified by direct calculation. The next property is the linearity of the divergence operator:

$$\nabla \cdot A = \nabla \cdot A_1 e_r + \nabla \cdot A_2 e_\theta + \nabla \cdot A_3 e_\phi.$$

By linearity, we will first compute $\nabla \cdot (A_1 e_r)$. This becomes

$$\begin{aligned}
\nabla \cdot (A_1 e_r) &= \nabla \cdot (A_1 h_2 h_3 [\frac{e_\theta}{h_2} \times \frac{e_\phi}{h_3}]) \\
&= \nabla (A_1 h_2 h_3) \cdot (\frac{e_\theta}{h_2} \times \frac{e_\phi}{h_3}) + A_1 h_2 h_3 \nabla \cdot [\frac{e_\theta}{h_2} \times \frac{e_\phi}{h_3}] \\
&= \nabla (A_1 h_2 h_3) \cdot (\frac{e_\theta}{h_2} \times \frac{e_\phi}{h_3}) \\
&= \frac{1}{h_2 h_3} e_r \cdot \nabla (A_1 h_2 h_3) \\
&= \frac{1}{h_1 h_2 h_3} \frac{\partial}{\partial r} (A_1 h_2 h_3).
\end{aligned}$$

In the above computation, we used the fact that for any two smooth functions f and g,

$$\nabla \cdot (\nabla f \times \nabla g) = 0.$$

By doing similar computations for $\nabla \cdot (A_2 e_\theta)$ and $\nabla \cdot (A_3 e_\phi)$ and adding our results together, we find that the divergence of a smooth vector field is

$$\nabla \cdot A = \frac{1}{r^2 \sin\theta} (\frac{\partial}{\partial r} (A_1 r^2 \sin\theta) + \frac{\partial}{\partial \theta} (A_2 r \sin\theta) + \frac{\partial}{\partial \phi} (A_3 r))$$

Now, we can finally compute the Laplacian as follows:

$$\begin{aligned}
\Delta f &= \nabla \cdot (\nabla f) \\
&= \nabla \cdot (\frac{\partial f}{\partial r} e_r + \frac{1}{r} \frac{\partial f}{\partial \theta} e_\theta + \frac{1}{r \sin\theta} \frac{\partial f}{\partial \phi} e_\phi) \\
&= \frac{1}{r^2 \sin\theta} (\frac{\partial}{\partial r} (\frac{\partial f}{\partial r} r^2 \sin\theta) + \frac{\partial}{\partial \theta} (\frac{1}{r} \frac{\partial f}{\partial \theta} r \sin\theta) + \frac{\partial}{\partial \phi} (\frac{1}{r \sin\theta} \frac{\partial f}{\partial \phi} r)).
\end{aligned}$$

The above expression can be written more simply as

$$\Delta f = \frac{1}{r^2 \sin\theta} (\frac{\partial}{\partial r} (r^2 \sin\theta \frac{\partial f}{\partial r}) + \frac{\partial}{\partial \theta} (\sin\theta \frac{\partial f}{\partial \theta}) + \frac{\partial}{\partial \phi} (\frac{1}{\sin\theta} \frac{\partial f}{\partial \phi})).$$

We can further simplify the above equation as follows:

$$\begin{aligned}
\Delta f &= \frac{1}{r^2\sin\theta}(\frac{\partial}{\partial r}(r^2\sin\theta\frac{\partial f}{\partial r}) + \frac{\partial}{\partial\theta}(\sin\theta\frac{\partial f}{\partial\theta}) + \frac{\partial}{\partial\phi}(\frac{1}{\sin\theta}\frac{\partial f}{\partial\phi})) \\
&= \frac{1}{r^2\sin\theta}(\sin\theta[r^2\frac{\partial^2 f}{\partial r^2} + 2r\frac{\partial f}{\partial r}] + \frac{\partial}{\partial\theta}(\sin\theta\frac{\partial f}{\partial\theta}) + \frac{1}{\sin\theta}\frac{\partial^2 f}{\partial\phi^2}) \\
&= \frac{\partial^2 f}{\partial r^2} + \frac{2}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial f}{\partial\theta}) + \frac{1}{\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}] \\
&= \frac{\partial^2 f}{\partial r^2} + \frac{2}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\Delta_{S^2} f \\
&= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial f}{\partial r}) + \frac{1}{r^2}\Delta_{S^2} f.
\end{aligned}$$

Note that this holds on $\mathbb{R}^3\backslash\{0\}$ since $r > 0$.

□

0.4 Question 4 (L4-4)

Part (a): We will state the question asked in part (a) as a lemma below.

Lemma 0.4.1. *Let $f : U \to End_{\mathbb{C}}(V)$ be a function, where $U \subseteq \mathbb{R}^n$ is open and V is a $\mathbb{C}$-vector space with dimension d. Let β be a basis for V. Then, f is smooth with respect to β if and only if it is smooth with respect to any basis for V.*

Proof. Assume that U and V are defined as above. Assume that β is a basis for V. Assume that f is defined as above. Note that if f is smooth with respect to any basis for V, then it must be smooth with respect to β. It suffices to prove the converse of this statement.

To show: (a) If f is smooth with respect to the basis β, then it is smooth with respect to any basis.

(a) Assume that f is smooth with respect to the basis β. Then, the following composite must be smooth:

$$U \xrightarrow{f} End_{\mathbb{C}}(V) \xrightarrow{C_\beta} M_{d\times d}(\mathbb{C}) \xrightarrow{\cong} \mathbb{C}^{d^2}$$

Let $[C]_\beta^\alpha$ denote the change of basis matrix from β to the basis α. Furthermore, let $\xi_\alpha : M_{d\times d}(\mathbb{C}) \to M_{d\times d}(\mathbb{C})$ be the map defined by

$$\xi_\alpha(A) = [C]_\beta^\alpha A([C]_\beta^\alpha)^{-1}$$

Then, consider the following composite:

$$U \xrightarrow{f} End_{\mathbb{C}}(V) \xrightarrow{C_\beta} M_{d\times d}(\mathbb{C}) \xrightarrow{\xi_\alpha} M_{d\times d}(\mathbb{C}) \xrightarrow{\cong} \mathbb{C}^{d^2}$$

The effect of this composite is to write the matrix representation of $f(u)$ in terms of the basis α, for all $u \in U$. In order to show that the composite above is smooth, it suffices to show that ξ_α is a smooth map. Assume that $A \in M_{d\times d}(\mathbb{C})$ so that

$$\xi_\alpha = [C]_\beta^\alpha A([C]_\beta^\alpha)^{-1}.$$

By identifying $M_{d\times d}(\mathbb{C})$ with $\mathbb{C}^{d^2}$, we observe that due to the matrix multiplication in ξ_α, every entry of $[C]_\beta^\alpha A([C]_\beta^\alpha)^{-1}$ is a polynomial of the entries in A. Since polynomials are smooth functions, we deduce that ξ_α must be a smooth function for all bases α of V. So, the above composite must be smooth, as it is a composite of smooth maps. Hence, f is smooth with respect to any basis α of V.

$\square$

Part (b): Our proof of the statement in part (b) will use the Leibnitz rule, which was asked in part (d).

Lemma 0.4.2. *Let $U \subseteq \mathbb{R}^n$ be an open set. Let β be a basis for V. For all $i \in \{1, \dots, n\}$ we define the $\mathbb{C}$-linear operator $\partial/\partial x_i$ on $C^\infty(U, End_{\mathbb{C}}(V))$ to be the following composite:*

$$C^\infty(U, End_{\mathbb{C}}(V)) \xrightarrow{C_\beta \circ (-)} C^\infty(U, M_{d\times d}(\mathbb{C})) \xrightarrow{\frac{\partial}{\partial x_i}} C^\infty(U, M_{d\times d}(\mathbb{C})) \xrightarrow{C_\beta^{-1} \circ (-)} C^\infty(U, End_{\mathbb{C}}(V))$$

The operator $\partial/\partial x_i$ acts on the matrices entry-wise. If $f : U \to M_{d\times d}(\mathbb{C})$ is identified with the matrix $(f(u))_{j,k}$, then the derivative is the matrix

$$(\frac{\partial f(u)}{\partial x_i})_{j,k}.$$

This operator is independent of the choice of basis β of V.

Proof. Assume that $\partial/\partial x_i$ is defined as above. Let α be another basis for V. We must show that the composite

$$C^\infty(U, End_{\mathbb{C}}(V)) \xrightarrow{C_\alpha \circ (-)} C^\infty(U, M_{d\times d}(\mathbb{C})) \xrightarrow{\frac{\partial}{\partial x_i}} C^\infty(U, M_{d\times d}(\mathbb{C})) \xrightarrow{C_\alpha^{-1} \circ (-)} C^\infty(U, End_{\mathbb{C}}(V))$$

gives the same result as the previous composite with basis β.

To show: (a) For all bases α of V, $\partial/\partial x_i$ must satisfy the Leibnitz rule. For all $f, g \in C^\infty(U, End_{\mathbb{C}}(V))$,

$$\frac{\partial}{\partial x_i}(fg) = \frac{\partial f}{\partial x_i} g + f \frac{\partial g}{\partial x_i}.$$

(a) Assume that $u \in U$. Suppose that $[f(u)]_\alpha, [g(u)]_\alpha \in M_{d\times d}(\mathbb{C})$ are the matrices of $f(u)$ and $g(u)$ respectively. That is, $C_\alpha(f(u)) = [f(u)]_\alpha$ and $C_\alpha(g(u)) = [g(u)]_\alpha$. Then, the matrix associated to $(fg)(u) = f(u) \circ g(u)$ is the product $[f(u)]_\alpha[g(u)]_\alpha$. The jk entry of this matrix product is

$$\sum_{l=1}^{d}([f(u)]_\alpha)_{jl}([g(u)]_\alpha)_{lk}.$$

Note that $([f(u)]_\alpha)_{jl}, ([g(u)]_\alpha)_{lk} \in \mathbb{C}$ for all $j, k, l \in \{1, \ldots, d\}$. Thus, when we differentiate the above expression, the Leibniz rule applies, resulting in the following expression (in tandem with $\mathbb{C}$-linearity of $\partial/\partial x_i$):

$$\sum_{l=1}^{d}\frac{\partial}{\partial x_i}(([f(u)]_\alpha)_{jl})([g(u)]_\alpha)_{lk} + ([f(u)]_\alpha)_{jl}\frac{\partial}{\partial x_i}(([g(u)]_\alpha)_{lk}).$$

Recognising the above expression as the sum of two products of two matrices, we can rewrite it as

$$\frac{\partial[f(u)]_\alpha}{\partial x_i}[g(u)]_\alpha + [f(u)]_\alpha\frac{\partial[g(u)]_\alpha}{\partial x_i}$$

Finally, when we apply C_α^{-1}, we obtain the smooth function

$$\frac{\partial f(u)}{\partial x_i}g(u) + f(u)\frac{\partial g(u)}{\partial x_i}.$$

Therefore,

$$\frac{\partial f(u)}{\partial x_i}g(u) + f(u)\frac{\partial g(u)}{\partial x_i} = \frac{\partial}{\partial x_i}((fg)(u)).$$

Since this holds for an arbitrary $u \in U$, we deduce that for all bases α of V,

$$\frac{\partial}{\partial x_i}(fg) = \frac{\partial f}{\partial x_i}g + f\frac{\partial g}{\partial x_i}.$$

Now we return to our original problem. Once again assume that $u \in U$ and $f \in C^\infty(U, End_{\mathbb{C}}(V))$. Then, the matrix of $f(u)$ with respect to the basis α can be expressed as

$$C_\alpha(f(u)) = [f(u)]_\alpha = [C]_\beta^\alpha[f(u)]_\beta([C]_\beta^\alpha)^{-1}.$$

Note that this means that

$$f(u) = C_\alpha^{-1}([C]_\beta^\alpha[f(u)]_\beta([C]_\beta^\alpha)^{-1}). \tag{1}$$

Applying $\partial/\partial x_i$ in conjunction with the Leibnitz rule proved in part (a), we deduce that

$$\begin{aligned}
\frac{\partial}{\partial x_i}([f(u)]_\alpha) &= \frac{\partial}{\partial x_i}([C]^\alpha_\beta [f(u)]_\beta ([C]^\alpha_\beta)^{-1}) \\
&= \frac{\partial}{\partial x_i}([C]^\alpha_\beta [f(u)]_\beta)([C]^\alpha_\beta)^{-1} + [C]^\alpha_\beta [f(u)]_\beta \frac{\partial}{\partial x_i}(([C]^\alpha_\beta)^{-1}) \quad \text{(Leibnitz Rule)} \\
&= \frac{\partial}{\partial x_i}([C]^\alpha_\beta [f(u)]_\beta)([C]^\alpha_\beta)^{-1} \\
&= (\frac{\partial}{\partial x_i}([C]^\alpha_\beta)[f(u)]_\beta + [C]^\alpha_\beta \frac{\partial}{\partial x_i}([f(u)]_\beta)])([C]^\alpha_\beta)^{-1} \quad \text{(Leibnitz Rule)} \\
&= [C]^\alpha_\beta \frac{\partial}{\partial x_i}([f(u)]_\beta)([C]^\alpha_\beta)^{-1}.
\end{aligned}$$

In the above calculation, we have repeatedly used the fact that the change of basis matrices $[C]^\alpha_\beta$ and $([C]^\alpha_\beta)^{-1}$ have constant entries. Thus, their derivatives are equal to the zero matrix. Now, we can apply C_α^{-1} to get

$$\begin{aligned}
C_\alpha^{-1}(\frac{\partial}{\partial x_i}([f(u)]_\alpha)) &= C_\alpha^{-1}([C]^\alpha_\beta \frac{\partial}{\partial x_i}([f(u)]_\beta)([C]^\alpha_\beta)^{-1}) \\
&= \frac{\partial f(u)}{\partial x_i} \quad \text{(From 1)} \\
&= C_\beta^{-1}(\frac{\partial}{\partial x_i}([f(u)]_\beta)).
\end{aligned}$$

Since this holds for all $u \in U$, we deduce from the above equality that the two composites for α and β are equal. So, $\partial/\partial x_i$ is independent of the choice of basis for V.

□

To reiterate, we have proved the statements in both parts (b) and (d) in the proof above.

Part (c): Here is statement we are required to prove.

Lemma 0.4.3. *Let $U \subseteq \mathbb{R}^n$ be open and $End_{\mathbb{C}}(V)$ denote the $\mathbb{C}$-vector space of linear operators on a finite-dimensional $\mathbb{C}$-vector space V. Let $C^\infty(U, End_{\mathbb{C}}(V))$ denote the space of smooth functions $f : U \to End_{\mathbb{C}}(V)$. Then, $C^\infty(U, End_{\mathbb{C}}(V))$ is a $\mathbb{C}$-algebra, with $(fg)(u) = f(u) \circ g(u)$.*

Proof. Assume that U is an open subset of $\mathbb{R}^n$, V is a finite-dimensional $\mathbb{C}$-vector space and $End_{\mathbb{C}}(V)$ is the $\mathbb{C}$-vector space of linear operators on V. Suppose that V has dimension d.

To show: (a) For all $f, g, h \in C^\infty(U, End_{\mathbb{C}}(V))$, $f(gh) = (fg)h$.

(b) There exists a function $1 \in C^\infty(U, End_{\mathbb{C}}(V))$ such that $1f = f1 = f$ for all $f \in C^\infty(U, End_{\mathbb{C}}(V))$.

(c) For all $f, g, h \in C^\infty(U, End_{\mathbb{C}}(V))$, $f(g + h) = fg + fh$.

(d) For all $f, g, h \in C^\infty(U, End_{\mathbb{C}}(V))$, $(g + h)f = gf + hf$.

(e) For all $f, g \in C^\infty(U, End_{\mathbb{C}}(V))$ and $\lambda \in \mathbb{C}$, $(\lambda f)g = f(\lambda g) = \lambda fg$.

(f) $C^\infty(U, End_{\mathbb{C}}(V))$ is a vector space with addition and scalar multiplication.

(a) Assume that $f, g, h \in C^\infty(U, End_{\mathbb{C}}(V))$. Then, a quick calculation shows that for all $u \in U$ and $v \in V$,

$$\begin{aligned} f(gh)(u)(v) &= (f(u) \circ (gh)(u))(v) \\ &= f(u)((gh)(u)(v)) \\ &= f(u)((g(u) \circ h(u))(v)) \\ &= f(u)(g(u)(h(u)(v))) \\ &= (f(u) \circ g(u))(h(u)(v)) \\ &= ((fg)(u) \circ h(u))(v) \\ &= (fg)h(u)(v). \end{aligned}$$

So, $f(gh) = (fg)h$.

(b) Define the function $1 : U \to End_{\mathbb{C}}(V)$ which sends $u \in U$ to id_V, where id_V is the identity map on V. To see that 1 is a smooth function, pick an ordered basis β of V and consider the composite Φ defined below:

$$U \xrightarrow{1} End_{\mathbb{C}}(V) \xrightarrow{C_\beta} M_{d \times d}(\mathbb{C}) \xrightarrow{\cong} \mathbb{C}^{d^2}$$

This composite sends any $u \in U$ to the $d \times d$ identity matrix I_d. Hence, Φ is smooth because each component function of Φ is a constant (since all of the entries of I_d are constant) and is thus, smooth. Furthermore, for all $u \in U$ and $v \in V$,

$$(1f)(u)(v) = (1(u) \circ f(u))(v) = (id_V \circ f(u))(v) = f(u)(v)$$

and

$$(f1)(u)(v) = (f(u) \circ id_V)(v) = f(u)(v).$$

This shows that $f1 = 1f = f$ for all $f \in C^\infty(U, End_{\mathbb{C}}(V))$.

(c) We compute for all $u \in U$ and $v \in V$ as follows:

$$\begin{aligned} f(g+h)(u)(v) &= (f(u) \circ (g+h)(u))(v) \\ &= f(u)((g+h)(u)(v)) \\ &= f(u)(g(u)(v) + h(u)(v)) \\ &= f(u)(g(u)(v)) + f(u)(h(u)(v)) \quad \text{since } f \text{ is a linear operator} \\ &= (f(u) \circ g(u))(v) + (f(u) \circ h(u))(v) \\ &= ((f \circ g) + (f \circ h))(u)(v). \end{aligned}$$

So, $f(g+h) = fg + fh$.

(d) We compute for all $u \in U$ and $v \in V$ as follows:

$$\begin{aligned} (g+h)f(u)(v) &= ((g+h)(u) \circ f(u))(v) \\ &= (g+h)(u)(f(u)(v)) \\ &= g(u)(f(u)(v)) + h(u)(f(u)(v)) \\ &= (g(u) \circ f(u))(v) + (h(u) \circ f(u))(v) \\ &= (gf + hf)(u)(v). \end{aligned}$$

Therefore, $(g+h)f = gf + hf$.

(e) Assume that $\lambda \in \mathbb{C}$. Then, we compute for all $u \in U$ and $v \in V$,

$$
\begin{aligned}
(\lambda f)g(u)(v) &= ((\lambda f)(u) \circ g(u))(v) \\
&= (\lambda f)(u)(g(u)(v)) \\
&= \lambda f(u)(g(u)(v)) \\
&= \lambda(f(u) \circ g(u))(v) \\
&= \lambda f g(u)(v)
\end{aligned}
$$

and

$$
\begin{aligned}
f(\lambda g)(u)(v) &= (f(u) \circ (\lambda g)(u))(v) \\
&= f(u)((\lambda g)(u)(v)) \\
&= f(u)(\lambda g(u)(v)) \\
&= \lambda f(u)(g(u)(v)) \\
&= \lambda(f(u) \circ g(u))(v) \\
&= \lambda f g(u)(v).
\end{aligned}
$$

So, $(\lambda f)g = f(\lambda g) = \lambda f g$.

(f) To see that $C^\infty(U, End_{\mathbb{C}}(V))$ is a vector space, we first note that it is closed under addition and scalar multiplication since adding two smooth functions results in a smooth function and multiplying a smooth function by a scalar results in a smooth function. Since $End_{\mathbb{C}}(V)$ is a $\mathbb{C}$-vector space, the vector space axioms must be satisfied for $f(u)$ for all $u \in U$ and $f \in C^\infty(U, End_{\mathbb{C}}(V))$. Since these results hold for all $u \in U$, it must hold for all f. So, $C^\infty(U, End_{\mathbb{C}}(V))$ must be a $\mathbb{C}$-vector space.

Thus, $C^\infty(U, End_{\mathbb{C}}(V))$ is a $\mathbb{C}$-algebra.

□

Part (e): The problem asked here is stated and proved below:

Lemma 0.4.4. *Let V be a $\mathbb{C}$-vector space with $\dim V = d$. Let $f \in C^\infty(U, End_{\mathbb{C}}(V))$, where U is an open subset of $\mathbb{R}^n$. Then, the functions $u \mapsto Tr(f(u))$ and $u \mapsto \det(f(u))$ are also smooth functions from U to $\mathbb{C}$.*

Proof. Assume that V is a $\mathbb{C}$-vector space with $\dim V = d$. Assume that $U \subseteq \mathbb{R}^n$ is open and $f \in C^\infty(U, End_{\mathbb{C}}(V))$. Recall that this means that the composite

$$U \xrightarrow{f} End_{\mathbb{C}}(V) \xrightarrow{C_\beta} M_{d\times d}(\mathbb{C}) \xrightarrow{\cong} \mathbb{C}^{d^2}$$

is smooth for any basis β of V. For the basis β of V, we define the trace function $T_\beta : U \to \mathbb{C}$ to be the composite

$$U \xrightarrow{f} End_{\mathbb{C}}(V) \xrightarrow{C_\beta} M_{d\times d}(\mathbb{C}) \xrightarrow{Tr} \mathbb{C}.$$

Similarly, we define the determinant function $D_\beta : U \to \mathbb{C}$ to be the composite

$$U \xrightarrow{f} End_{\mathbb{C}}(V) \xrightarrow{C_\beta} M_{d\times d}(\mathbb{C}) \xrightarrow{\det} \mathbb{C}.$$

We will first show that the functions T_β and D_β are independent of the choice of basis for V.

To show: (a) If α and β are two different choices of basis for V, then $T_\beta = T_\alpha$.

(b) If α and β are two different choices of basis for V, then $D_\beta = D_\alpha$.

(a) Assume that α is another basis for V and that T_α is the composite below:

$$U \xrightarrow{f} End_{\mathbb{C}}(V) \xrightarrow{C_\alpha} M_{d\times d}(\mathbb{C}) \xrightarrow{Tr} \mathbb{C}.$$

We must show that for all $u \in U$, $T_\beta(u) = T_\alpha(u)$. Using the composite, the LHS evaluates as $T_\beta(u) = Tr([f(u)]_\beta)$. The RHS evaluates as

$$T_\alpha(u) = Tr([f(u)]_\alpha) = Tr([C]^\alpha_\beta [f(u)]_\beta ([C]^\alpha_\beta)^{-1}).$$

We will require the preliminary result below in order to proceed further.

To show: (aa) If $A, B \in M_{d\times d}(\mathbb{C})$, then $Tr(AB) = Tr(BA)$.

(aa) Assume that $A, B \in M_{d\times d}(\mathbb{C})$. Let $A = (a_{ij})$ and $B = (b_{ij})$. Then,

$$Tr(AB) = \sum_{i=1}^{d} \sum_{k=1}^{d} a_{ik} b_{ki}.$$

We can exploit the commutativity of addition and multiplication in $\mathbb{C}$ in order to rewrite the above expression as

$$
\begin{aligned}
Tr(AB) &= \sum_{i=1}^{d}\sum_{k=1}^{d} a_{ik}b_{ki} \\
&= \sum_{i=1}^{d}\sum_{k=1}^{d} b_{ki}a_{ik} \\
&= \sum_{i=1}^{d}(b_{1i}a_{i1} + \cdots + b_{di}a_{id}) \\
&= (b_{11}a_{11} + \cdots + b_{d1}a_{1d}) + (b_{12}a_{21} + \cdots + b_{d2}a_{2d}) + \cdots + (b_{1d}a_{d1} + \cdots + b_{dd}a_{dd}) \\
&= (b_{11}a_{11} + b_{12}a_{21} + \cdots + b_{1d}a_{d1}) + \cdots + (b_{d1}a_{1d} + b_{d2}a_{2d} + \cdots + b_{dd}a_{dd}) \\
&= \sum_{k=1}^{d}\sum_{i=1}^{d} b_{ki}a_{ik} \\
&= Tr(BA).
\end{aligned}
$$

So, $Tr(AB) = Tr(BA)$.

(a) Now, we can use the result in part (aa) to deduce that

$$
\begin{aligned}
T_\alpha(u) &= Tr([C]_\beta^\alpha [f(u)]_\beta ([C]_\beta^\alpha)^{-1}) \\
&= Tr(([C]_\beta^\alpha [f(u)]_\beta)([C]_\beta^\alpha)^{-1}) \\
&= Tr(([C]_\beta^\alpha)^{-1}([C]_\beta^\alpha [f(u)]_\beta)) \quad \text{(Part (aa))} \\
&= Tr([f(u)]_\beta) \\
&= T_\beta(u).
\end{aligned}
$$

Since $u \in U$ was arbitrary, we deduce that $T_\alpha = T_\beta$.

(b) From the definition, $D_\alpha : U \to \mathbb{C}$ is the composite

$$
U \xrightarrow{f} End_{\mathbb{C}}(V) \xrightarrow{C_\alpha} M_{d\times d}(\mathbb{C}) \xrightarrow{\det} \mathbb{C}.
$$

We want to show that for all $u \in U$, $D_\beta(u) = D_\alpha(u)$. Applying the appropriate composite, the LHS is $\det([f(u)]_\beta)$, whereas the RHS is

$$
D_\alpha(u) = \det([f(u)]_\alpha) = \det([C]_\beta^\alpha [f(u)]_\beta ([C]_\beta^\alpha)^{-1}).
$$

However, by the multiplicative property of the determinant,

$$\begin{aligned} D_\alpha(u) &= \det([C]_\beta^\alpha [f(u)]_\beta ([C]_\beta^\alpha)^{-1}) \\ &= \det([C]_\beta^\alpha) \det([f(u)]_\beta) \det(([C]_\beta^\alpha)^{-1})) \\ &= \det([C]_\beta^\alpha) \det(([C]_\beta^\alpha)^{-1})) \det([f(u)]_\beta) \\ &= \det([f(u)]_\beta) \\ &= D_\beta(u). \end{aligned}$$

Since $u \in U$ was arbitrary, we deduce that $D_\alpha = D_\beta$.

Because the definitions of D_β and T_β are independent of the choice of basis β for V, we will now call these composites D and T.

To show: (c) D is a smooth function.

(d) T is a smooth function.

(c) From the definition of D, it suffices to show that the determinant map $\det : M_{d\times d}(\mathbb{C}) \to \mathbb{C}$ is smooth. First, we identify $M_{d\times d}(\mathbb{C})$ with $\mathbb{C}^{d^2}$. Then, for all $A \in \mathbb{C}^{d^2}$, $\det(A)$ is a polynomial of the entries in A. Since polynomials are smooth, $\det : \mathbb{C}^{d^2} \to \mathbb{C}$ must be a smooth function. Returning to the definition of D as the composite

$$U \xrightarrow{f} End_{\mathbb{C}}(V) \xrightarrow{C_\beta} M_{d\times d}(\mathbb{C}) \xrightarrow{\det} \mathbb{C}$$

we observe that since f is smooth, the composite $C_\beta \circ f : U \to \mathbb{C}^{d^2}$ must be smooth. So, D is the composite of smooth maps and is thus, smooth.

(d) Correspondingly to part (c), it suffices to show that the trace map $Tr : M_{d\times d}(\mathbb{C}) \to \mathbb{C}$ is smooth. But, for all $A \in M_{d\times d}(\mathbb{C}) \cong \mathbb{C}^{d^2}$, $Tr(A)$ is a polynomial of the entries in A. Since polynomials are smooth, $Tr : \mathbb{C}^{d^2} \to \mathbb{C}$ must be a smooth function. Hence, $T : U \to \mathbb{C}$ is a smooth function because it is the composite of the smooth maps $C_\beta \circ f$ and Tr:

$$U \xrightarrow{f} End_{\mathbb{C}}(V) \xrightarrow{C_\beta} M_{d\times d}(\mathbb{C}) \xrightarrow{Tr} \mathbb{C}$$

□

0.5 Question 5 (L4-6)

We are given the equation

$$1 = \rho(R^x_{2\pi}) = \exp(2\pi[x_2 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_2}]).$$

Recall that the expression $x_2 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_2}$ appeared as an infinitesimal generator of the symmetry $\rho(R^x_\alpha)$. It was found in lectures that

$$\frac{d}{d\alpha}(\rho(R^x_\alpha))|_{\alpha=0} = x_2 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_2}.$$

Since $\rho(R^x_{2\pi})$ is the identity operator on $\mathcal{P}_k(3)$, denoted by 1 in the first equation, we would like to identify $\rho(R^x_{2\pi})$ with the 3×3 identity matrix in $SO(3)$, which we will denote by I.

In order to simplify the situation, we would also like to identify the infinitesimal $x_2 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_2}$ with a 3×3 matrix. In order to do this, we will repeat the derivation of this differential operator, but with R^x_α rather than its representation $\rho(R^x_\alpha)$:

$$\begin{aligned} \frac{d}{d\alpha}(R^x_\alpha)|_{\alpha=0} &= \frac{d}{d\alpha}\begin{pmatrix} 1 & & \\ & \cos\alpha & -\sin\alpha \\ & \sin\alpha & \cos\alpha \end{pmatrix}|_{\alpha=0} \\ &= \begin{pmatrix} 1 & & \\ & -\sin\alpha & -\cos\alpha \\ & \cos\alpha & -\sin\alpha \end{pmatrix}|_{\alpha=0} \\ &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}. \end{aligned}$$

So, we can take the above matrix to be the 3×3 matrix representation of $x_2 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_2}$. Let

$$X = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

The equation we originally have is equivalent to $I = \exp(2\pi X)$. The trick to revealing the trigonometric identity underpinning this identity is to realise that

$$X^3 = -X$$

From this, we deduce that for all $n \in \mathbb{Z}_{>0}$,

$$X^{2n+2} = (-1)^n X^2 \text{ and } X^{2n+1} = (-1)^n X.$$

Now, for a general $\alpha \in \mathbb{R}$, we argue as follows

$$\begin{aligned}
\rho(R^x_\alpha) &= \exp(\alpha X) \\
&= \sum_{i=0}^{\infty} \frac{(\alpha X)^i}{i!} \\
&= I + \sum_{i=1}^{\infty} \frac{(\alpha X)^i}{i!} \\
&= I + \sum_{i=0}^{\infty} \frac{\alpha^{2i+1} X^{2i+1}}{(2i+1)!} + \sum_{i=1}^{\infty} \frac{\alpha^{2i} X^{2i}}{(2i)!} \\
&= I + \sum_{i=0}^{\infty} \frac{\alpha^{2i+1}(-1)^i X}{(2i+1)!} + \sum_{i=1}^{\infty} \frac{\alpha^{2i}(-1)^{i+1} X^2}{(2i)!} \\
&= I + \sin(\alpha) X + (1 - \cos(\alpha)) X^2.
\end{aligned}$$

By substituting $\alpha = 2\pi$, we recover the original matrix identity $I = \exp(2\pi X)$.

0.6 Question 6 (L4-8)

Our first step is to understand the operators $\sigma(R^x_\alpha)$, $\sigma(R^y_\alpha)$ and $\sigma(R^z_\alpha)$ as exponentials of differential operators in Cartesian coordinates. We already know from the lectures that

$$\sigma(R^x_\alpha) = \exp(\alpha[y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}])$$

as an operator on $\mathcal{H}_k(S^2)$. Note that $\alpha \in \mathbb{R}$. We will derive the analogous expressions for $\sigma(R^y_\alpha)$ and $\sigma(R^z_\alpha)$.

Consider the operator $\rho(R^y_\alpha) : \mathcal{P}_k(3) \to \mathcal{P}_k(3)$. Let $\{x^\beta \mid |\beta| = k\}$ be a $\mathbb{C}$-basis for $\mathcal{P}_k(3)$. The derivative of the operator with respect to α can be computed as

$$\begin{aligned}
\frac{d}{d\alpha}(\rho(R^y_\alpha))(x^\beta) &= \frac{d}{d\alpha}(\varphi_{R^y_\alpha}(x^\beta)) \\
&= \frac{d}{d\alpha}[(x_1\cos\alpha + x_3\sin\alpha)^{\beta_1}x_2^{\beta_2}(-x_1\sin\alpha + x_3\cos\alpha)^{\beta_3}] \\
&= (-x_1\sin\alpha + x_3\cos\alpha)\beta_1(x_1\cos\alpha + x_3\sin\alpha)^{\beta_1-1}x_2^{\beta_2}(-x_1\sin\alpha + x_3\cos\alpha)^{\beta_3} \\
&\quad +(x_1\cos\alpha + x_3\sin\alpha)^{\beta_1}[x_2^{\beta_2}(-x_1\cos\alpha - x_3\sin\alpha) \\
&\quad \beta_3(-x_1\sin\alpha + x_3\cos\alpha)^{\beta_3-1}] \\
&= \beta_1 x_2^{\beta_2}(x_1\cos\alpha + x_3\sin\alpha)^{\beta_1-1}(-x_1\sin\alpha + x_3\cos\alpha)^{\beta_3+1} \\
&\quad -\beta_3 x_2^{\beta_2}(-x_1\sin\alpha + x_3\cos\alpha)^{\beta_3-1}(x_1\cos\alpha + x_3\sin\alpha)^{\beta_1+1} \\
&= \varphi_{R^y_\alpha}([x_3\frac{\partial}{\partial x_1} - x_1\frac{\partial}{\partial x_3}](x^\beta))
\end{aligned}$$

Hence,

$$\frac{d}{d\alpha}(\rho(R^y_\alpha)) = \rho(R^y_\alpha) \circ [x_3\frac{\partial}{\partial x_1} - x_1\frac{\partial}{\partial x_3}].$$

Utilising a similar procedure to the lectures and then transferring over to the operator $\sigma(R^y_\alpha)$ on $\mathcal{H}_k(S^2)$, we obtain

$$\sigma(R^y_\alpha) = \exp(\alpha[z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}]).$$

Here, we map x_1 to x, x_2 to y and x_3 to z. The derivation of the exponential expression for $\sigma(R^z_\alpha)$ is very similar. First, we compute that

$$\begin{aligned}
\frac{d}{d\alpha}(\rho(R^z_\alpha))(x^\beta) &= \frac{d}{d\alpha}(\varphi_{R^z_\alpha}(x^\beta)) \\
&= \frac{d}{d\alpha}[(x_1\cos\alpha - x_2\sin\alpha)^{\beta_1}(x_1\sin\alpha + x_2\cos\alpha)^{\beta_2}x_3^{\beta_3}] \\
&= (-x_1\sin\alpha - x_2\cos\alpha)\beta_1(x_1\cos\alpha - x_2\sin\alpha)^{\beta_1-1}(x_1\sin\alpha + x_2\cos\alpha)^{\beta_2}x_3^{\beta_3} \\
&\quad +(x_1\cos\alpha - x_2\sin\alpha)^{\beta_1}x_3^{\beta_3}\beta_2(x_1\cos\alpha - x_2\sin\alpha)(x_1\sin\alpha + x_2\cos\alpha)^{\beta_2-1} \\
&= -\beta_1 x_3^{\beta_3}(x_1\cos\alpha - x_2\sin\alpha)^{\beta_1-1}(x_1\sin\alpha + x_2\cos\alpha)^{\beta_2+1} \\
&\quad +\beta_2 x_3^{\beta_3}(x_1\cos\alpha - x_2\sin\alpha)^{\beta_1+1}(x_1\sin\alpha + x_2\cos\alpha)^{\beta_2-1} \\
&= \varphi_{R^z_\alpha}([x_1\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_1}](x^\beta)).
\end{aligned}$$

Therefore,

$$\sigma(R^z_\alpha) = \exp(\alpha[x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}]).$$

Now, we will covert all of the Cartesian differential operators to differential operators in spherical coordinates. First, express the conversion of Cartesian coordinates to spherical coordinates as matrices:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r\cos\phi\sin\theta \\ r\sin\phi\sin\theta \\ r\cos\theta \end{pmatrix}.$$

Here $\phi \in [0, 2\pi]$ and $\theta \in [0, \pi]$. We can use the chain rule to express the matrix $[dx, dy, dz]^T$ as

$$\begin{aligned}
\begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} &= \begin{pmatrix} \cos\phi\sin\theta dr + r\cos\phi\cos\theta d\theta - r\sin\phi\sin\theta d\phi \\ \sin\phi\sin\theta dr + r\sin\phi\cos\theta d\theta + r\cos\phi\sin\theta d\phi \\ \cos\theta dr - r\sin\theta d\theta \end{pmatrix}. \\
&= \begin{pmatrix} \cos\phi\sin\theta & r\cos\phi\cos\theta & -r\sin\phi\sin\theta \\ \sin\phi\sin\theta & r\sin\phi\cos\theta & r\cos\phi\sin\theta \\ \cos\theta & -r\sin\theta & 0 \end{pmatrix} \begin{pmatrix} dr \\ d\theta \\ d\phi \end{pmatrix}
\end{aligned}$$

We can invert this matrix to solve for $[dr, d\theta, d\phi]^T$:

$$\begin{pmatrix} dr \\ d\theta \\ d\phi \end{pmatrix} = \begin{pmatrix} \cos\phi\sin\theta & \sin\theta\sin\phi & \cos\theta \\ \frac{\cos\theta\cos\phi}{r} & \frac{\sin\phi\cos\theta}{r} & -\frac{\sin\theta}{r} \\ -\frac{\sin\phi}{r\sin\theta} & \frac{\cos\phi}{r\sin\theta} & 0 \end{pmatrix} \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}.$$

By reading off the columns of the square matrix, we deduce that

$$\frac{\partial}{\partial x} = \cos\phi\sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi}$$

$$\frac{\partial}{\partial y} = \sin\theta\sin\phi\frac{\partial}{\partial r} + \frac{\sin\phi\cos\theta}{r}\frac{\partial}{\partial\theta} + \frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial\phi}$$

and

$$\frac{\partial}{\partial z} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}.$$

It remains to substitute the above expressions and the spherical coordinates for x, y and z into our expressions for $\sigma(R^x_\alpha), \sigma(R^y_\alpha)$ and $\sigma(R^z_\alpha)$. Once we do this, we obtain the following:

$$\sigma(R^x_\alpha) = \exp(\alpha[-\sin\phi\frac{\partial}{\partial\theta} - \cos\phi\cot\theta\frac{\partial}{\partial\phi}])$$

$$\sigma(R^y_\alpha) = \exp(\alpha[\cos\phi\frac{\partial}{\partial\theta} - \sin\phi\cot\theta\frac{\partial}{\partial\phi}])$$

and

$$\sigma(R^z_\alpha) = \exp(\alpha\frac{\partial}{\partial\phi}).$$

Now we will turn to the general case and write a formula for $\sigma(R^{\widehat{n}}_\alpha)$ as an exponential, where $\widehat{n} \in S^2$ is an arbitrary unit vector. In order to do this, we will modify spherical coordinates so that $\widehat{n}$ plays the role of the x axis in usual spherical coordinates.

Recall that in $\mathcal{P}_k(3)$, if $\widehat{n} = R^z_\varphi R^y_{\eta-\pi/2}(e_1)$, then the variables (t_1, t_2, t_3) are related to (x_1, x_2, x_3) by

$$\begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix} = R^z_\varphi R^y_{\eta-\pi/2} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

Here, $\varphi \in [0, 2\pi]$ and $\eta \in [0, \pi]$. If we were to create spherical coordinates where $\widehat{n}$ plays the role of the x axis, then we expect the variable r to remain unchanged (since rotations are isometric), whereas the angles θ and ϕ in usual spherical coordinates are offset by angles of $\eta - \pi/2$ and φ respectively. Define

1. $x' = r\sin(\theta' + \eta - \frac{\pi}{2})\cos(\phi' + \varphi) = -r\cos(\theta' + \eta)\cos(\phi' + \varphi)$

2. $y' = r\sin(\theta' + \eta - \frac{\pi}{2})\sin(\phi' + \varphi) = -r\cos(\theta' + \eta)\sin(\phi' + \varphi)$

3. $z' = r\cos(\theta' + \eta - \frac{\pi}{2}) = r\sin(\theta' + \eta)$

where $\theta' \in [0, \pi]$ and $\phi' \in [0, 2\pi]$. As a consistency check, if we substitute $r = 1$, $\theta' = \pi/2$ and $\phi' = 0$ into the above coordinates, we obtain

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \sin\eta\cos\varphi \\ \sin\eta\sin\varphi \\ \cos\eta \end{pmatrix} = R^z_\varphi R^y_{\eta-\pi/2}(e_1) = \widehat{n}.$$

Thus, in our new coordinate system, $\widehat{n}$ is the new x axis, which is what we want. By converting from an operator in $\mathcal{P}_k(3)$ to an operator in $\mathcal{H}_k(S^2)$, we deduce that

$$\sigma(R^{\widehat{n}}_\alpha) = \exp(\alpha[y'\frac{\partial}{\partial z'} - z'\frac{\partial}{\partial y'}])$$

where under the isomorphism from $\mathcal{H}_k(3)$ to $\mathcal{H}_k(S^2)$, $t_1 \mapsto x'$, $t_2 \mapsto y'$ and $t_3 \mapsto z'$. It remains to write the differential operators $\frac{\partial}{\partial z'}$ and $\frac{\partial}{\partial y'}$ in our new spherical coordinate system. We will approach this in a similar manner to the previous case.

By the chain rule,

$$\begin{pmatrix} dx' \\ dy' \\ dz' \end{pmatrix} = \begin{pmatrix} -\cos(\theta'+\eta)\cos(\phi'+\varphi) & r\sin(\theta'+\eta)\cos(\phi'+\varphi) & r\cos(\theta'+\eta)\sin(\phi'+\varphi) \\ -\cos(\theta'+\eta)\sin(\phi'+\varphi) & r\sin(\theta'+\eta)\sin(\phi'+\varphi) & -r\cos(\theta'+\eta)\cos(\phi'+\varphi) \\ \sin(\theta'+\eta) & r\cos(\theta'+\eta) & 0 \end{pmatrix}$$

$$\cdot \begin{pmatrix} dr \\ d\theta' \\ d\phi' \end{pmatrix}.$$

Inverting the Jacobian again, we find that

$$\frac{\partial}{\partial x'} = -\cos(\varphi+\phi')\cos(\theta'+\eta)\frac{\partial}{\partial r} + \frac{\cos(\varphi+\phi')\sin(\theta'+\eta)}{r}\frac{\partial}{\partial \theta'} + \frac{\sin(\varphi+\phi')}{r\cos(\theta'+\eta)}\frac{\partial}{\partial \phi'}$$

$$\frac{\partial}{\partial y'} = -\sin(\varphi+\phi')\cos(\theta'+\eta)\frac{\partial}{\partial r} + \frac{\sin(\varphi+\phi')\sin(\theta'+\eta)}{r}\frac{\partial}{\partial \theta'} - \frac{\cos(\varphi+\phi')}{r\cos(\theta'+\eta)}\frac{\partial}{\partial \phi'}$$

and

$$\frac{\partial}{\partial z'} = \sin(\theta' + \eta)\frac{\partial}{\partial r} + \frac{\cos(\theta' + \eta)}{r}\frac{\partial}{\partial \theta'}.$$

By making the appropriate substitutions, we obtain

$$y'\frac{\partial}{\partial z'} - z'\frac{\partial}{\partial y'} = -\sin(\phi' + \varphi)\frac{\partial}{\partial \theta'} + \cos(\phi' + \varphi)\tan(\theta' + \eta)\frac{\partial}{\partial \phi'}.$$

Hence,

$$\sigma(R_{\alpha}^{\widehat{n}}) = \exp(\alpha[\cos(\phi' + \varphi)\tan(\theta' + \eta)\frac{\partial}{\partial \phi'} - \sin(\phi' + \varphi)\frac{\partial}{\partial \theta'}]).$$