Lie Algebras, Assignment 1

Jackson Godfrey [redacted]

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1 Ex L2-2

Let $U : \mathcal{H} \to \mathcal{H}$ be a linear unitary transformation. Suppose that $U(x) = U(y)$, fix $w \in \mathcal{H}$ arbitrarily. Observe

$$0 = \langle 0, U(w) \rangle = \langle U(x) - U(y), U(w) \rangle = \langle U(x-y), U(w) \rangle = \langle x - y, w \rangle$$

Since $\langle x-y, w \rangle = 0$ for all w, it must be true that $x - y = 0$. So $x = y$ and U is injective.

Consider the hilbert space l^2 of square-summable sequences with inner product

$$\langle (z_n)_n, (w_n)_n \rangle := \sum_{n=1}^{\infty} z_n \overline{w_n}$$

The mapping $F : l^2 \to l^2$ given by

$$(a_1, a_2, \dots) \mapsto (0, a_1, a_2, \dots)$$

is clearly linear and not surjective. Also F is unitary because

$$\langle F((z)_n), F((w)_n) \rangle = 0 + \sum_{i=2}^{\infty} z_{i-1} \overline{w_{i-1}} = \sum_{i=1}^{\infty} z_i \overline{w_i} = \langle (z)_n, (w)_n \rangle$$

2 Ex L2-4

Let U be either the unitary+linear or anti-unitary+anti-linear transformation as defined in the proof of Wigner's theorem. Let $\phi \in \mathcal{H}$. Since W is dense then we can write $\phi = \lim_{n\to\infty} \phi_n$ with $\phi_n \in W$. Then $U^{ext}(\phi)$ is defined as $\lim_{n\to\infty}(U(\phi_n))$. This is well-defined, which I assume I do not need to prove. Regardless of what property U has, additivity is shown by

$$\begin{aligned} U^{ext}(\phi+\psi) &= \lim_{n\to\infty} U(\phi_n+\psi_n) \\ &= \lim_{n\to\infty} U(\phi_n)+U(\psi_n) \\ &= \lim_{n\to\infty} U(\phi_n)+\lim_{n\to\infty} U(\psi_n) = U^{ext}(\phi)+U^{ext}(\psi) \end{aligned}$$

We could exchange the limit with the + operation precisely because $+ : \mathbb{C}\times\mathbb{C}\to\mathbb{C}$ is continuous. Here implicitly we used the fact that if $\phi = \lim_{n\to\infty}\phi_n$ and $\psi = \lim_{n\to\infty}\psi_n$ then $\phi+\psi = \lim_{n\to\infty}\phi_n+\psi_n$ Now if U is anti-linear we have:

$$U^{ext}(\lambda\phi) = \lim_{n\to\infty} U(\lambda\phi) = \lim_{n\to\infty} \overline{\lambda}\cdot U(\phi) = \overline{\lambda}\lim_{n\to\infty} U(\phi) = \overline{\lambda}U^{ext}(\phi)$$

The limit commutes with scalar multiplication, that being a continuous map $\mathbb{C}\times\mathbb{C}\to\mathbb{C}$. Also implicitly above we used $\lim_{n\to\infty}\lambda\phi = \lambda\lim_{n\to\infty}\phi_n$. The unitary case is similar, in particular there is no need to conjugate λ above. Still assuming that U is anti-linear and anti-unitary, we show finally that U^{ext} is anti-unitary:

$$\begin{aligned} \langle U^{ext}(\phi), U^{ext}(\psi)\rangle &= \langle \lim_{n\to\infty} U(\phi_n), \lim_{m\to\infty} U(\psi_n)\rangle \\ &= \lim_{n\to\infty}\lim_{m\to\infty}\langle U(\phi_n), U(\psi_m)\rangle \qquad \text{Since } \langle -,-\rangle \text{ is cts} \\ &= \lim_{n\to\infty}\lim_{m\to\infty}\overline{\langle \phi_n,\psi_m\rangle} \qquad U \text{ is anti-unitary} \\ &= \lim_{n\to\infty}\lim_{m\to\infty}\langle \psi_m,\phi_n\rangle \\ &= \langle \lim_{m\to\infty}\psi_m, \lim_{m\to\infty}\phi_n\rangle \\ &= \langle \psi,\phi\rangle \\ &= \overline{\langle \phi,\psi\rangle} \end{aligned}$$

The case where U is linear and unitary is less verbose but ultimately the same.

Now to show that U^{ext} is surjective. Since U^{ext} is either linear and unitary or anti-linear and anti-unitary, it is automatically injective, so bijectivity follows from surjectivity. Let $\{\psi_k\}$ be the orthonormal dense basis for $\mathcal{H}$ as in the proof of Wigner's theorem. Then U is defined as follows on unit vectors $\phi = \sum_{k=1} C_k\psi_k$ with $C_1 \neq 0$:

$$\begin{aligned} U(\phi) &= \sum_{k=1} C_k U(\psi_k) \quad \text{unitary case} \\ U(\phi) &= \sum_{k=1} \overline{C_k} U(\psi_k) \quad \text{anti-unitary case} \end{aligned}$$

U is defined on arbitrary vectors by extending this definition in the obvious way. Now

$$W' := \{\phi' \in \mathcal{H} \mid \langle U(\psi_1), \phi'\rangle \neq 0\}$$

is a dense subset of $\mathcal{H}$ for the exact same reasons that W is. Let $\phi' \in \mathcal{H}$. Then by the above discussion we can write $\phi' = \lim_{n\to\infty}\phi'_n$ where $\phi'_n \in W'$. Let $\mathcal{S}'_k$ be the ray containing ϕ'_k, since Q is surjective $Q(\mathcal{S}_k) = \mathcal{S}'_k$ for some ray $\mathcal{S}_k$ for all k. Let $\phi_k \in \mathcal{S}_k$ be chosen arbitrarily. Now observe that

$$\begin{aligned} |\langle \psi_1, \phi_k \rangle| &= (\mathcal{R}_1, \mathcal{S}_k) \qquad\qquad \mathcal{R}_1 \text{ being the ray containing } \psi_1 \\ &= (Q(\mathcal{R}_1), Q(\mathcal{S}_k)) \\ &= |\langle U(\psi_1), \phi'_k \rangle| \neq 0 \qquad \text{since } \phi'_k \in W' \end{aligned}$$

So we see that $\phi_k \in W$ for all k. Therefore we can conclude that $U(\phi_k) \in Q(\mathcal{S}_k) = S'_k$, so for all k there exists $\lambda_k \in U(1)$ such that

$$\lambda_k \cdot U(\phi_k) = \phi'_k$$

In the unitary case we conclude that

$$\phi = \lim_{n\to\infty} \phi'_k = \lim_{n\to\infty} U(\lambda_k \phi_k) = U^{ext}(\lim_{k\to\infty} \lambda_k \phi_k)$$

In the anti-unitary case we conclude that

$$\phi = \lim_{n\to\infty} \phi'_k = \lim_{n\to\infty} U(\overline{\lambda_k} \phi_k) = U^{ext}(\lim_{k\to\infty} \overline{\lambda_k} \phi_k)$$

So U^{ext} is surjective. Now, I am not sure I *need* to do this, but just in case: a proof that $\lambda_k \phi_k$ is a cauchy sequence:

$$\begin{aligned} \|\lambda_n \phi_n - \lambda_m \phi_m\| &= \langle \lambda_n \phi_n - \lambda_m \phi_m, \lambda_n \phi_n - \lambda_m \phi_m \rangle \\ &= \langle U(\lambda_n \phi_n - \lambda_m \phi_m), U(\lambda_n \phi_n - \lambda_m \phi_m) \rangle \qquad \text{unitary} \\ &= \langle \lambda_n U(\phi_n) - \lambda_m U(\phi_m), \lambda_n U(\phi_n) - \lambda_m U(\phi_m) \rangle \quad \text{by linearity} \\ &= \|\lambda_n U(\phi_n) - \lambda_m U(\phi_m)\| \\ &= \|\phi'_n - \phi'_m\| \end{aligned}$$

So cauchyness follows from that of $\{\phi'_n\}$. In the anti-unitary case the same process will show that $\overline{\lambda_k} \phi_k$ is cauchy as well.