Exercises:

6.1. Show that if A is torsionfree, $\mathrm{Ext}(A,\mathbb{Z})$ is divisible, and that if A is divisible, $\mathrm{Ext}(A,\mathbb{Z})$ is torsionfree. Show conversely that if $\mathrm{Ext}(A,\mathbb{Z})$ is divisible, A is torsionfree and that if $\mathrm{Ext}(A,\mathbb{Z})$ is torsionfree and $\mathrm{Hom}(A,\mathbb{Z})=0$ then A is divisible. (See Exercise 5.8.)

6.2. Show that $\mathrm{Ext}(\mathbb{Q},\mathbb{Z})$ is divisible and torsionfree, and hence a $\mathbb{Q}$-vector space. (Compare Exercise 2.4.) Deduce that $\mathrm{Ext}(\mathbb{Q},\mathbb{Z})\cong\mathbb{R}$, $\mathrm{Hom}(\mathbb{Q},\mathbb{Q}/\mathbb{Z})\cong\mathbb{R}$. Compute $\mathrm{Ext}(\mathbb{R},\mathbb{Z})$.

6.3. Show that $\mathrm{Ext}(\mathbb{Q}/\mathbb{Z},\mathbb{Z})$ fits into exact sequences

$$0\to\mathbb{Z}\to\mathrm{Ext}(\mathbb{Q}/\mathbb{Z},\mathbb{Z})\to\mathbb{R}\to 0,$$

$$0\to\mathrm{Ext}(\mathbb{Q}/\mathbb{Z},\mathbb{Z})\to\mathbb{R}\to\mathbb{Q}/\mathbb{Z}\to 0.$$

6.4. Show that the simultaneous equations $\mathrm{Ext}(A,\mathbb{Z})=0$, $\mathrm{Hom}(A,\mathbb{Z})=0$ imply $A=0$.

6.5. Show that the simultaneous equations $\mathrm{Ext}(A,\mathbb{Z})=\mathbb{Q}$, $\mathrm{Hom}(A,\mathbb{Z})=0$ have no solution. Generalize this by replacing $\mathbb{Q}$ by a suitable $\mathbb{Q}$-vector space. What can you say of the solutions of $\mathrm{Ext}(A,\mathbb{Z})=\mathbb{R}$, $\mathrm{Hom}(A,\mathbb{Z})=0$?

7. The Tensor Product

In the remaining two sections of Chapter III we shall introduce two functors: the tensor product and the Tor-functor.

Let Λ again be a ring, A a right and B a left Λ-module.

Definition. The *tensor product* of A and B over Λ is the abelian group, $A\otimes_\Lambda B$, obtained as the quotient of the free abelian group on the set of all symbols $a\otimes b$, $a\in A$, $b\in B$, by the subgroup generated by

$$(a_1+a_2)\otimes b-(a_1\otimes b+a_2\otimes b),\ a_1,a_2\in A, b\in B;$$

$$a\otimes(b_1+b_2)-(a\otimes b_1+a\otimes b_2),\ a\in A, b_1,b_2\in B;$$

$$a\lambda\otimes b-a\otimes\lambda b,\ a\in A, b\in B, \lambda\in\Lambda.$$

In case $\Lambda=\mathbb{Z}$ we shall allow ourselves to write $A\otimes B$ for $A\otimes_{\mathbb{Z}}B$. For simplicity we shall denote the element of $A\otimes_\Lambda B$ obtained as canonical image of $a\otimes b$ in the free abelian group by the same symbol $a\otimes b$.

The ring Λ may be regarded as left or right Λ-module over Λ. It is easy to see that we have natural isomorphisms (of abelian groups)

$$\Lambda\otimes_\Lambda B\xrightarrow{\sim}B,\quad A\otimes_\Lambda\Lambda\xrightarrow{\sim}A$$

given by $\lambda\otimes b\mapsto\lambda b$ and $a\otimes\lambda\mapsto a\lambda$.

For any $\alpha: A\to A'$ we define an induced map $\alpha_*: A\otimes_\Lambda B\to A'\otimes_\Lambda B$ by $\alpha_*(a\otimes b)=(\alpha a)\otimes b$, $a\in A$, $b\in B$. Also, for $\beta: B\to B'$ we define

$\beta_* : A \otimes_\Lambda B \to A \otimes_\Lambda B'$ by $\beta_*(a \otimes b) = a \otimes (\beta b)$, $a \in A$, $b \in B$. With these definitions we obtain

Proposition 7.1. *For any left Λ-module B, $- \otimes_\Lambda B : \mathfrak{M}^r_\Lambda \to \mathfrak{Ab}$ is a covariant functor. For any right Λ-module A, $A \otimes_\Lambda - : \mathfrak{M}^l_\Lambda \to \mathfrak{Ab}$ is a covariant functor. Moreover, $- \otimes_\Lambda -$ is a bifunctor.*

The proof is left to the reader. □

If $\alpha : A \to A'$ and $\beta : B \to B'$ are homomorphisms we use the notation

$$\alpha \otimes \beta = \alpha_* \beta_* = \beta_* \alpha_* : A \otimes_\Lambda B \to A' \otimes_\Lambda B'.$$

The importance of the tensorproduct will become clear from the following assertion.

Theorem 7.2. *For any right Λ-module A, the functor $A \otimes_\Lambda - : \mathfrak{M}^l_\Lambda \to \mathfrak{Ab}$ is left adjoint to the functor $\mathrm{Hom}_{\mathbb{Z}}(A, -) : \mathfrak{Ab} \to \mathfrak{M}^l_\Lambda$.*

Proof. The left-module structure of $\mathrm{Hom}_{\mathbb{Z}}(A, -)$ is induced by the right-module structure of A (see Section I.8). We have to show that there is a natural transformation η such that for any abelian group G and any left Λ-module B

$$\eta : \mathrm{Hom}_{\mathbb{Z}}(A \otimes_\Lambda B, G) \xrightarrow{\sim} \mathrm{Hom}_\Lambda(B, \mathrm{Hom}_{\mathbb{Z}}(A, G)).$$

Given $\varphi : A \otimes_\Lambda B \to G$ we define $\eta(\varphi)$ by the formula

$$((\eta(\varphi))(b))(a) = \varphi(a \otimes b).$$

Given $\psi : B \to \mathrm{Hom}_{\mathbb{Z}}(A, G)$ we define $\tilde{\eta}(\psi)$ by $(\tilde{\eta}(\psi))(a \otimes b) = (\psi(b))(a)$. We claim that $\eta, \tilde{\eta}$ are natural homomorphisms which are inverse to each other. We leave it to the reader to check the necessary details. □

Analogously we may prove that $- \otimes_\Lambda B : \mathfrak{M}^r_\Lambda \to \mathfrak{Ab}$ is left adjoint to $\mathrm{Hom}_{\mathbb{Z}}(B, -) : \mathfrak{Ab} \to \mathfrak{M}^r_\Lambda$, where the right module structure of $\mathrm{Hom}_{\mathbb{Z}}(B, G)$ is given by the left module structure of B. We remark that the tensor-product-functor $A \otimes_\Lambda -$ is determined up to natural equivalence by the adjointness property of Theorem 7.2 (see Proposition II.7.3); a similar remark applies to the functor $- \otimes_\Lambda B$.

As an immediate consequence of Theorem 7.2 we have

Proposition 7.3. (i) *Let $\{B_j\}$, $j \in J$, be a family of left Λ-modules and let A be a right Λ-module. Then there is a natural isomorphism*

$$A \otimes_\Lambda \Big(\bigoplus_{j \in J} B_j\Big) \xrightarrow{\sim} \bigoplus_{j \in J} (A \otimes_\Lambda B_j).$$

(ii) *If $B' \xrightarrow{\beta'} B \xrightarrow{\beta''} B'' \to 0$ is an exact sequence of left Λ-modules, then for any right Λ-module A, the sequence*

$$A \otimes_\Lambda B' \xrightarrow{\beta'_*} A \otimes_\Lambda B \xrightarrow{\beta''_*} A \otimes_\Lambda B'' \to 0$$

is exact.

Proof. By the dual of Theorem II.7.7 a functor possessing a right adjoint preserves coproducts and cokernels. □

Of course there is a proposition analogous to Proposition 7.2 about the functor $-\otimes_\Lambda B$ for fixed B. The reader should note that, even if β' in Proposition 7.3 (ii) is monomorphic, β'_* will not be monomorphic in general: Let $\Lambda = \mathbb{Z}$, $A = \mathbb{Z}_2$, and consider the exact sequence $\mathbb{Z} \overset{\mu}{\rightarrowtail} \mathbb{Z} \twoheadrightarrow \mathbb{Z}_2$ where μ is multiplication by 2. Then

$$\mu_*(n \otimes m) = n \otimes 2m = 2n \otimes m = 0 \otimes m = 0,$$

$n \in \mathbb{Z}_2$, $m \in \mathbb{Z}$. Hence $\mu_*: \mathbb{Z}_2 \otimes \mathbb{Z} \to \mathbb{Z}_2 \otimes \mathbb{Z}$ is the zero map, while $\mathbb{Z}_2 \otimes \mathbb{Z} \cong \mathbb{Z}_2$.

Definition. A left Λ-module B is called *flat* if for every short exact sequence $A' \overset{\mu}{\rightarrowtail} A \overset{\varepsilon}{\twoheadrightarrow} A''$ the induced sequence

$$0 \to A' \otimes_\Lambda B \xrightarrow{\mu_*} A \otimes_\Lambda B \to A'' \otimes_\Lambda B \to 0$$

is exact. This is to say that for every monomorphism $\mu: A' \to A$ the induced homomorphism $\mu_*: A' \otimes_\Lambda B \to A \otimes_\Lambda B$ is a monomorphism. also.

Proposition 7.4. *Every projective module is flat.*

Proof. A projective module P is a direct summand in a free module. Hence, since $A \otimes_\Lambda -$ preserves sums, it suffices to show that free modules are flat. By the same argument it suffices to show that Λ as a left module is flat. But this is trivial since $A \otimes_\Lambda \Lambda \cong A$. □

For abelian groups it turns out that "flat" is "torsionfree" (see Exercise 8.7). Since the additive group of the rationals $\mathbb{Q}$ is torsionfree but not free, one sees that flat modules are not, in general, projective.

Exercises:

7.1. Show that if A is a left Γ-right Λ-bimodule and B a left Λ-right Σ-bimodule then $A \otimes_\Lambda B$ may be given a left Γ-right Σ-bimodule structure.

7.2. Show that, if Λ is commutative, we can speak of the tensorproduct $A \otimes_\Lambda B$ of two left (!) Λ-modules, and that $A \otimes_\Lambda B$ has an obvious Λ-module structure. Also show that then $A \otimes_\Lambda B \cong B \otimes_\Lambda A$ and $(A \otimes_\Lambda B) \otimes_\Lambda C \cong A \otimes_\Lambda (B \otimes_\Lambda C)$ by canonical isomorphisms.

7.3. Prove the following generalization of Theorem 7.2. Let A be a left Γ-right Λ-bimodule, B a left Λ-module and C a left Γ-module. Then $A \otimes_\Lambda B$ can be given a left Γ-module structure, and $\mathrm{Hom}_\Gamma(A, C)$ a left Λ-module structure. Prove the adjointness relation

$$\eta: \mathrm{Hom}_\Gamma(A \otimes_\Lambda B, C) \xrightarrow{\sim} \mathrm{Hom}_\Lambda(B, \mathrm{Hom}_\Gamma(A, C)).$$

7.4. Show that, if A, B are Λ-modules and if $\sum_i a_i \otimes b_i = 0$ in $A \otimes_\Lambda B$, then there are finitely generated submodules $A_0 \subseteq A$, $B_0 \subseteq B$ such that $a_i \in A_0$, $b_i \in B_0$ and $\sum_i a_i \otimes b_i = 0$ in $A_0 \otimes B_0$.