Recall that last lecture we defined a functor from topological spaces to simplicial sets

$$S: \underline{Top} \longrightarrow \underline{SSef}$$
 (1.1)

on objects by defining SX to be the functor  $SX = h_{X} \cdot (\Delta)^{\circ p}$ , as in

 $\mathbb{A}^{p} \xrightarrow{(\Delta^{p})^{p}} \xrightarrow{\text{Top}^{p}} \xrightarrow{h_{X}} \underbrace{\text{Set}} (1.2)$ 

where  $\Delta^{\bullet}: \mathbb{A} \longrightarrow \mathbb{Top}$  was the standard simplices  $[n] \mapsto \Delta^{n} \subseteq \mathbb{R}^{n+1}$ . Given continuous  $\mathcal{Y}: X \longrightarrow Y$  we defined a natural transformation  $S\mathcal{Y}: SX \longrightarrow SY$ "by hand" and in today's lecture we re-examine this more abstractly via (1.2).

Lemma Let  $C, P, P_2$  be categories with C small and let  $Q: P \longrightarrow P_2$  be a functor. Then there is an induced functor

$$A_{q} : [C, R] \longrightarrow [C, R]$$

$$A_{q}(F) = Q \circ F$$

$$\int A_{q}(\alpha: F \longrightarrow F') : Q \circ F \longrightarrow Q \circ F'$$

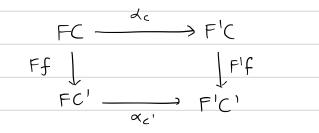
$$A_{q}(\alpha)_{c} = Q(\alpha_{c})$$

Proof Finture need to check 
$$A_Q(A)$$
, as defined, is a natural transformation  
 $Q \circ F \longrightarrow Q \circ F'$ . So let  $f: C \longrightarrow C'$  be a morphism in  $C$ . We have to  
check that  
 $A_Q(A)_C$   
 $(Q \circ F)(C) \longrightarrow (Q \circ F')(C)$   
 $(Q \circ F)(f) \int (Q \circ F)(f)$   
 $(Q \circ F)(C') \longrightarrow (Q \circ F')(C')$ 

 $(\mathbf{I})$ 

(2)

commutes. But this is the image under Q of the diagram



which commutes by naturality of  $\alpha$ , so we we done:  $A\alpha(\alpha)$  is a natural transformation  $Q \circ F \longrightarrow Q \circ F'$  and thus a morphism in [C, 8.].

So we have defined a mapping  $A \otimes on objects and morphisms$ , it remains to check that it is functorial:  $A \otimes (id: F \longrightarrow F)_c = \otimes (id_c) = id_{\otimes c}$  so  $A \otimes (id) = id$ , and

$A_{\alpha}(g \circ f)_{c} = Q((g \circ f)_{c})$	(def N Aa)
$= Q(g_c \circ f_c)$	(dup N composition of nat. trans)
$= Q(g_c) \circ Q(f_c)$	(Q is a functor)
$= A_{\alpha}(g)_{c} \circ A_{\alpha}(f)_{c}$	
$= \int A_{\alpha}(9) \circ A_{\alpha}(f) \int_{C}$	

Hence  $A_{\alpha}(g \circ f) = A_{\alpha}(g) \circ A_{\alpha}(f)$  and we've done.  $\prod$ 

<u>Lemma</u> Let C, Cz, P be categories with C, Cz small and let  $Q: C, \longrightarrow Cz$ be a functor. Then there is an induced functor  $B_Q: [C_2, P] \longrightarrow [C_1, P]$ defined by  $B_Q(F) = F \circ Q$ ,  $B_Q(A: F \longrightarrow F')_C = \mathcal{A}_{QC}$ .

Ex 1 Prove the Lemma.

Example From  $\Delta^{\bullet}: \Lambda \longrightarrow \text{Top}$  we get a functor  $(\Delta^{\bullet})^{\circ P}: \Lambda^{\circ P} \longrightarrow \text{Top}^{\circ P}$  and thus

$$\begin{bmatrix} \text{Top}^{\circ p}, \text{ set} \end{bmatrix} \longrightarrow \begin{bmatrix} \mathbb{N}^{\circ p}, \text{ set} \end{bmatrix} = \underline{\text{Sset}}$$

$$F \xrightarrow{} F \xrightarrow{} F \xrightarrow{} (\mathbb{N}^{\circ})^{\circ p}$$

(modulo Top not being small. Either ignore this or use a small, full subcategory  $C \subseteq Top$  including all the  $\Delta^{r's}$ ). The next step is to construct a functor as marked by the dotted line in:

$$\underline{\text{Top}} \xrightarrow{} - - - - \rightarrow [\underline{\text{Top}}^{\circ p}, \underline{\text{fet}}] \xrightarrow{} (\underline{\Lambda}^{\circ p}, \underline{\text{fet}}] = \underline{\text{sfet}}$$

$$\chi \longmapsto h_{\chi} \longmapsto h_{\chi} \circ (\underline{\Lambda}^{\circ})^{\circ p} =: SX$$

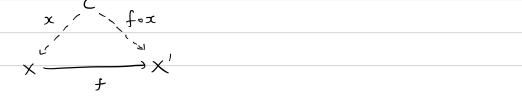
Lemma Let  $\mathcal{C}$  be a small category. There is a functor  $h_{(-)} : \mathcal{C} \longrightarrow [\mathcal{C}^{op}, jet]$ defined on objects by  $X \longmapsto h_X$  and on morphisms  $f : X \longrightarrow X'$  by The natural transformation

$$h_{f}: h_{X} \longrightarrow h_{X'}$$

$$(h_{f})_{c}: h_{X}(c) \longrightarrow h_{x'}(c)$$

$$(h_{f})_{c}: h_{X}(c) \longrightarrow h_{x'}(c)$$

$$(h_{f})_{c}(x) := f \circ x$$



<u>Proof</u>  $\bigcirc$  <u>hf</u> is a natural transformation of function hx, hx':  $\mathbb{C}^{\circ P} \longrightarrow \underline{\text{fet}}$ 

Suppose 
$$t: C \rightarrow C$$
 in  $\mathcal{C}$ , thus  $C \rightarrow C'$  in  $\mathcal{C}^{sp}$ . We need to show  
commutativity of the order square in  

$$h_{\mathbf{x}}(c) \xrightarrow{h_{\mathbf{x}}(t)} h_{\mathbf{x}}(c')$$

$$\begin{array}{c} h_{\mathbf{x}}(c) \xrightarrow{h_{\mathbf{x}}(t)} f_{\mathbf{x}}(c') \xrightarrow{h_{\mathbf{x}}(c')} \xrightarrow{h_{\mathbf{x}}(c')} f_{\mathbf{x}}(c') \xrightarrow{h_{\mathbf{x}}(c')} \xrightarrow{h_{\mathbf{x}}(c')} f_{\mathbf{x}}(c') \xrightarrow{h_{\mathbf{x}}(c')} \xrightarrow{h_{\mathbf{x}}(c')} \xrightarrow{h_{\mathbf{x}}(c')} \xrightarrow{h_{\mathbf{x}}(c')} \xrightarrow{h_{\mathbf{x}}(c')} f_{\mathbf{x}}(c') \xrightarrow{h_{\mathbf{x}}(c')} \xrightarrow{h_{\mathbf{x}}(c'$$

(4)

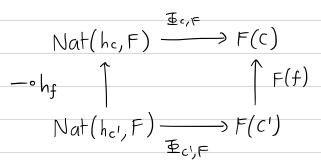
## $\underline{E \times 2}$ (Yoneda Lemma) Let C be a category, and $F: \mathcal{C}^{\circ p} \longrightarrow \underline{Jet}$ a functor. Writing

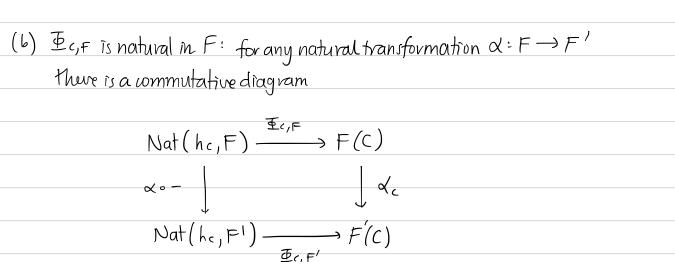
$$Nat(F, F') := \{ natural transformations F \rightarrow F' \}$$

prove that the map

$$\underline{\Phi}_{c,F}: \operatorname{Nat}(h_{c},F) \longrightarrow F(C), \quad \underline{\Phi}_{c,F}(\alpha) = \alpha_{c}(id_{c})$$

is a bijection with the following properties





(c) Is there a unique family { Ec, F}c, F of bijections satisfying the naturality in (a), (b)? Discuss.

(3)

