In this lecture we continue with examples of functor, and introduce the notion of a representing object for certain functor.

Example Let C be a category and A an object of C. There is a functor $\begin{array}{c} x \\ \beta \\ f \end{array} \xrightarrow{\mathcal{A}} B' \\ f \end{array}$ $h^A: \mathcal{G} \longrightarrow \underline{Set}$ • $h^{A}(B) = \mathcal{C}(A, B)$ $for f: B \rightarrow B', h^{A}(f): C(A, B) \rightarrow C(A, B')$ is defined by $h^{A}(f)(x) = f \circ x$ in the sense of p. D of Lecture 2 $Le \ \mathbb{Z}(M \times N) := V(M \times N)$ there Ex 1 Check h^A is a functor. Example Let M, N be abelian groups, and recall that their tensor product MON is defined to be the quotient of the free abelian group Z(M×N) generated by the set $R = \left\{ (m_1 + m_2, n) - (m_3, n) - (m_2, n) \right\}_{m_1, m_2 \in M, n \in N} U$ $\left\{ \left(m, n_1 + n_2 \right) - \left(m, n_1 \right) - \left(m, n_2 \right) \right\} m \in M, n_1, n_2 \in N$ Let u denote the composite (Las in Lecture 2) $M \times N \xrightarrow{l} \mathbb{Z}(M \times N) \longrightarrow \mathbb{Z}(M \times N) / \langle R \rangle =: M \otimes N$ $1.e. \quad m \otimes n := \mathcal{M}(m, n).$

> This is a definition by "construction". Arguably more natural is the following style of definition "universal properly + existence proof". The point is that mis the <u>universal</u> bilinear form on M×N.

Example Consider the functor

$$F: \underline{Ab} \longrightarrow \underline{Set}$$

•
$$F(T) = \begin{cases} \alpha: M \times N \rightarrow T \mid \alpha \text{ is bilinear, i.e.} \\ \alpha(m_1 + m_{2_1}n) = \alpha(m_1,n) + \alpha(m_{2_1}n) \text{ and} \\ \alpha(m_1,n_{1_2}) = \alpha(m_1,n) + \alpha(m_{2_1}n) \text{ and} \\ \alpha(m_1,n_{1_2}) = \alpha(m_1,n) + \alpha(m_{2_1}n) \text{ for} \\ \alpha(m_1,m_{2_1}m \in M \text{ and } n_1,n_{2_1}n \in N) \end{cases}$$
•
$$F(\beta:T \rightarrow T') : F(T) \longrightarrow F(T') \text{ is defined by} \\ \text{sending } \alpha: M \times N \rightarrow T \text{ bs } \beta \circ \alpha: M \times N \rightarrow T'. \end{cases}$$
Lemma There is for every abelian group T a bijection
$$\Phi_T : \text{ Hom}_{Ab}(M \otimes N, T) \longrightarrow F(T) \qquad (2.1)$$
defined by $\Phi_T(\sigma) = \mathcal{T} \circ \mathcal{M}$. Moreover this bijection is $natural_1$, i.e. for any morphism $\beta: T \rightarrow T'$ the following diagram commutes
$$\Phi_T : Hom_{Ab}(M \otimes N, T) \longrightarrow F(T) \qquad (2.2)$$

Proof By construction
$$M : M \neq N \rightarrow M \otimes N$$
 is bilinear. We claim it is
universal, i.e. that if $d:M \neq N \rightarrow T$ is some other bilinear form
then there is a unique $\widehat{a}: M \otimes N \rightarrow T$ such that
 $(morphism in Ab)$

2

commuter The existence of \tilde{a} is clear: bilinearity of Δ means $\hat{\alpha}(R)=0$, ro $\hat{\lambda}$ factors via $\mathbb{Z}(M \times N)/\langle R \rangle = M \oplus N$, and we call this factorisation $\tilde{\alpha}$. By construction $\tilde{\alpha} \circ \mu = \Delta$, and $\tilde{\alpha}$ is unique making (3-1) commute (why?). We conclude that

$$\overline{\Phi}_{T}^{-1}(\alpha) = \widetilde{\alpha}$$

defines a two-sided inverse to $\overline{\Box}_{\tau}$, completing the proof (naturality is an easy exercise). \Box

$$h^{MON}(T) \xrightarrow{\cong} F(T).$$

We say there is a <u>natural isomorphism of functor</u> $h^{M \otimes N} \cong F$, and say that $(M \otimes N, M)$ is the solution of a <u>universal problem</u>, namely: what is the universal bilinear form on $M \times N$? This universal problem is encoded by F.

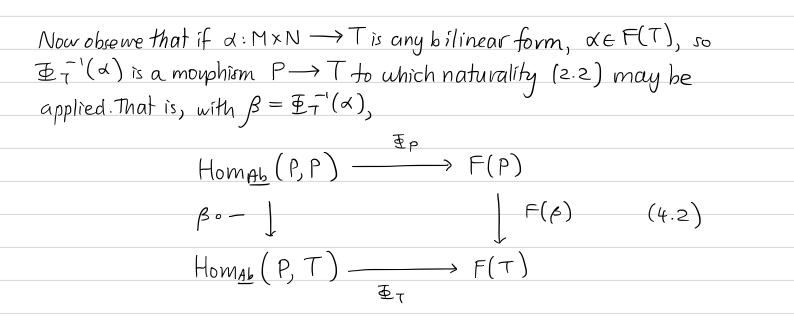
Note	We would like to make the connection between a family of bijections
	(2.1) which is natural (2.2) and the notion of a "universal bilinear
	form on M×N". Suppose to this end that we had never heard of the
	tensor puducl, but suppose we were given an abelian group P and
	a family of bijections, one for each TE Ab

$$\underline{\mathfrak{F}}_{\tau}: \operatorname{Hom}_{\underline{Ab}}(P, T) \xrightarrow{\cong} F(T) \qquad (4.1)$$

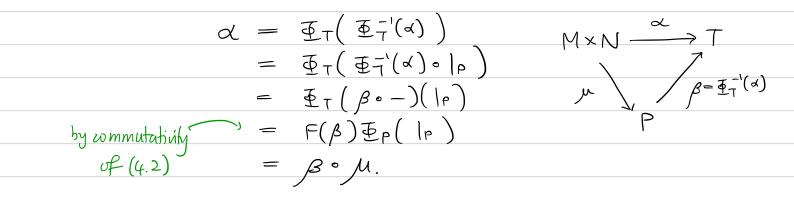
which are natural in the sense that the diagrams (2.2) commute for all β with P in place of MOON. Applying (4.1) to T=P we have

$$Hom_{\underline{Ab}}(P, P) \xrightarrow{\Phi_{P}} F(P)$$

 $\therefore \mu := \overline{\Phi}_P(1_P)$ is some bilinear form $\mu: M \times N \rightarrow P$.



commutes. But this means



That is, & factors through M. Moveover, this factorisation is unique:

5

commutes.