6.2. Give a universal characterization of kernel and cokernel, and show that kernel and cokernel are dual notions.

6.3. Dualize the assertions of Lemma 1.1, the Five Lemma (Exercise 1.2) and those of Exercises 3.4 and 3.5.

6.4. Let $\varphi : A \rightarrow B$. Characterize $\operatorname{im}\varphi$, $\varphi^{-1}B_0$ for $B_0 \subseteq B$, without using elements. What are their duals? Hence (or otherwise) characterize exactness.

6.5. What is the dual of the canonical homomorphism $\sigma : \bigoplus_{i\in J} A_i \rightarrow \prod_{i\in J} A_i$? What is the dual of the assertion that σ is an injection? Is the dual true?

7. Injective Modules over a Principal Ideal Domain

Recall that by Corollary 5.2 every projective module over a principal ideal domain is free. It is reasonable to expect that the injective modules over a principal ideal domain also have a simple structure. We first define:

Definition. Let Λ be an integral domain. A Λ-module D is *divisible* if for every $d \in D$ and every $0 \neq \lambda \in \Lambda$ there exists $c \in D$ such that $\lambda c = d$. Note that we do not require the uniqueness of c.

We list a few examples:

(a) As $\mathbb{Z}$-module the additive group of the rationals $\mathbb{Q}$ is divisible. In this example c is uniquely determined.

(b) As $\mathbb{Z}$-module $\mathbb{Q}/\mathbb{Z}$ is divisible. Here c is *not* uniquely determined.

(c) The additive group of the reals $\mathbb{R}$, as well as $\mathbb{R}/\mathbb{Z}$, are divisible.

(d) A non-trivial finitely generated abelian group A is never divisible. Indeed, A is a direct sum of cyclic groups, which clearly are not divisible.

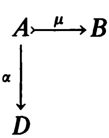
Theorem 7.1. *Let Λ be a principal ideal domain. A Λ-module is injective if and only if it is divisible.*

Proof. First suppose D is injective. Let $d \in D$ and $0 \neq \lambda \in \Lambda$. We have to show that there exists $c \in D$ such that $\lambda c = d$. Define $\alpha : \Lambda \rightarrow D$ by $\alpha(1) = d$ and $\mu : \Lambda \rightarrow \Lambda$ by $\mu(1) = \lambda$. Since Λ is an integral domain, $\mu(\xi) = \xi\lambda = 0$ if and only if $\xi = 0$. Hence μ is monomorphic. Since D is injective, there exists $\beta : \Lambda \rightarrow D$ such that $\beta\mu = \alpha$. We obtain

$$d = \alpha(1) = \beta\mu(1) = \beta(\lambda) = \lambda\beta(1).$$

Hence by setting $c = \beta(1)$ we obtain $d = \lambda c$. (Notice that so far no use is made of the fact that Λ is a principal ideal domain.)

Now suppose D is divisible. Consider the following diagram



We have to show the existence of $\beta: B \to D$ such that $\beta\mu = \alpha$. To simplify the notation we consider μ as an embedding of a submodule A into B. We look at pairs (A_j, α_j) with $A \subseteq A_j \subseteq B$, $\alpha_j: A_j \to D$ such that $\alpha_j|_A = \alpha$. Let Φ be the set of all such pairs. Clearly Φ is nonempty, since (A, α) is in Φ. The relation $(A_j, \alpha_j) \leqq (A_k, \alpha_k)$ if $A_j \subseteq A_k$ and $\alpha_k|_{A_j} = \alpha_j$ defines an ordering in Φ. With this ordering Φ is inductive. Indeed, every chain (A_j, α_j), $j \in J$ has an upper bound, namely $(\bigcup A_j, \bigcup \alpha_j)$ where $\bigcup A_j$ is simply the union, and $\bigcup \alpha_j$ is defined as follows: If $a \in \bigcup A_j$, then $a \in A_k$ for some $k \in J$. We define $\bigcup \alpha_j(a) = \alpha_k(a)$. Plainly $\bigcup \alpha_j$ is well-defined and is a homomorphism, and

$$(A_j, \alpha_j) \leqq (\bigcup A_j, \bigcup \alpha_j).$$

By Zorn's Lemma there exists a maximal element $(\bar{A}, \bar{\alpha})$ in Φ. We shall show that $\bar{A} = B$, thus proving the theorem. Suppose $\bar{A} \neq B$; then there exists $b \in B$ with $b \notin \bar{A}$. The set of $\lambda \in \Lambda$ such that $\lambda b \in \bar{A}$ is readily seen to be an ideal of Λ. Since Λ is a principal ideal domain, this ideal is generated by one element, say λ_0. If $\lambda_0 \neq 0$. then we use the fact that D is divisible to find $c \in D$ such that $\bar{\alpha}(\lambda_0 b) = \lambda_0 c$. If $\lambda_0 = 0$, we choose an arbitrary c. The homomorphism $\bar{\alpha}$ may now be extended to the module $\tilde{A}$ generated by $\bar{A}$ and b, by setting $\tilde{\alpha}(\bar{a} + \lambda b) = \bar{\alpha}(\bar{a}) + \lambda c$. We have to check that this definition is consistent. If $\lambda b \in \bar{A}$, we have $\tilde{\alpha}(\lambda b) = \lambda c$. But $\lambda = \xi \lambda_0$ for some $\xi \in \Lambda$ and therefore $\lambda b = \xi \lambda_0 b$. Hence

$$\bar{\alpha}(\lambda b) = \bar{\alpha}(\xi \lambda_0 b) = \xi \bar{\alpha}(\lambda_0 b) = \xi \lambda_0 c = \lambda c.$$

Since $(\bar{A}, \bar{\alpha}) < (\tilde{A}, \tilde{\alpha})$, this contradicts the maximality of $(\bar{A}, \bar{\alpha})$, so that $\bar{A} = B$ as desired. □

Proposition 7.2. *Every quotient of a divisible module is divisible.*

Proof. Let $\varepsilon: D \twoheadrightarrow E$ be an epimorphism and let D be divisible. For $e \in E$ and $0 \neq \lambda \in \Lambda$ there exists $d \in D$ with $\varepsilon(d) = e$ and $d' \in D$ with $\lambda d' = d$. Setting $e' = \varepsilon(d')$ we have $\lambda e' = \lambda \varepsilon(d') = \varepsilon(\lambda d') = \varepsilon(d) = e$. □

As a corollary we obtain the dual of Corollary 5.3.

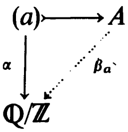
Corollary 7.3. *Let Λ be a principal ideal domain. Every quotient of an injective Λ-module is injective.* □

Next we restrict ourselves temporarily to abelian groups and prove in that special case

Proposition 7.4. *Every abelian group may be embedded in a divisible (hence injective) abelian group.*

The reader may compare this Proposition to Proposition 4.3, which says that every Λ-module is a quotient of a free, hence projective, Λ-module.

Proof. We shall define a monomorphism of the abelian group A into a direct product of copies of $\mathbb{Q}/\mathbb{Z}$. By Proposition 6.3 this will suffice. Let $0 \neq a \in A$ and let (a) denote the subgroup of A generated by a. Define $\alpha : (a) \to \mathbb{Q}/\mathbb{Z}$ as follows: If the order of $a \in A$ is infinite choose $0 \neq \alpha(a)$ arbitrary. If the order of $a \in A$ is finite, say n, choose $0 \neq \alpha(a)$ to have order dividing n. Since $\mathbb{Q}/\mathbb{Z}$ is injective, there exists a map $\beta_a : A \to \mathbb{Q}/\mathbb{Z}$ such that the diagram

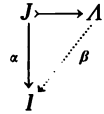


is commutative. By the universal property of the product, the β_a define a unique homomorphism $\beta : A \to \prod_{\substack{a \in A \\ a \neq 0}} (\mathbb{Q}/\mathbb{Z})_a$. Clearly β is a monomorphism since $\beta_a(a) \neq 0$ if $a \neq 0$. □

For abelian groups, the additive group of the integers $\mathbb{Z}$ is projective and has the property that to any abelian group $G \neq 0$ there exists a non-zero homomorphism $\varphi : \mathbb{Z} \to G$. The group $\mathbb{Q}/\mathbb{Z}$ has the dual properties; it is injective and to any abelian group $G \neq 0$ there is a nonzero homomorphism $\psi : G \to \mathbb{Q}/\mathbb{Z}$. Since a direct sum of copies of $\mathbb{Z}$ is called free, we shall term a direct product of copies of $\mathbb{Q}/\mathbb{Z}$ *cofree*. Note that the two properties of $\mathbb{Z}$ mentioned above do *not* characterize $\mathbb{Z}$ entirely. Therefore "cofree" is *not* the exact dual of "free", it is dual only in certain respects. In Section 8 the generalization of this concept to arbitrary rings is carried through.

Exercises:

7.1. Prove the following proposition: The Λ-module I is injective if and only if for every left ideal $J \subset \Lambda$ and for every Λ-module homomorphism $\alpha : J \to I$ the diagram



may be completed by a homomorphism $\beta : \Lambda \to I$ such that the resulting triangle is commutative. (Hint: Proceed as in the proof of Theorem 7.1.)

7.2. Let $0 \to R \to F \to A \to 0$ be a short exact sequence of abelian groups, with F free. By embedding F in a direct sum of copies of $\mathbb{Q}$, show how to embed A in a divisible group.

7.3. Show that every abelian group admits a unique maximal divisible subgroup.

7.4. Show that if A is a finite abelian group, then $\mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z}) \cong A$. Deduce that if there is a short exact sequence $0 \to A' \to A \to A'' \to 0$ of abelian groups with A finite, then there is a short exact sequence $0 \to A'' \to A \to A' \to 0$.

7.5. Show that a torsion-free divisible group D is a $\mathbb{Q}$-vector space. Show that $\mathrm{Hom}_{\mathbb{Z}}(A, D)$ is then also divisible. Is this true for any divisible group D?

7.6. Show that $\mathbb{Q}$ is a direct summand in a direct product of copies of $\mathbb{Q}/\mathbb{Z}$.