is a push-out. Show (i) that there exists $\omega: B'_1 \rightarrowtail B'$ such that $\omega\beta_1 = \beta, \omega\varphi'_1 = \varphi'$, and (ii) that the second square above is also a pull-back.

9.5. Let $\mathfrak{A}$ be an abelian category with arbitrary products and coproducts. Define the canonical sum-to-product morphism $\omega: \bigoplus_i A_i \to \prod_i A_i$, and prove that it is not true in general that ω is a monomorphism.

9.6. Let $\mathfrak{A}$ be an abelian category and $\mathfrak{C}$ a small category. Show that the functor category $\mathfrak{A}^{\mathfrak{C}}$ is also abelian. (Hint: Define kernels and cokernels componentwise).

9.7. Give examples of additive categories in which (i) not every morphism has a kernel, (ii) not every morphism has a cokernel.

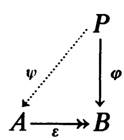
9.8. Prove Corollary 9.8. Give a counterexample in a non-abelian category.

10. Projective, Injective, and Free Objects

Although our interest in projective and injective objects is confined, in this book, to abelian categories, we will define them in an arbitrary category since the elementary results we adduce in this section will have nothing to do with abelian, or even additive, categories. Our principal purpose in including this short section is to clarify the categorical connection between freeness and projectivity. However, Proposition 10.2 will be applied in Section IV.12, and again later in the book.

The reader will recall the notion of projective and injective modules in Chapter I. Abstracting these notions to an arbitrary category, we are led to the following definitions.

Definition. An object P of a category $\mathfrak{C}$ is said to be *projective* if given the diagram



in $\mathfrak{C}$ with ε epimorphic, there exists ψ with $\varepsilon\psi = \varphi$. An object J of $\mathfrak{C}$ is said to be *injective* if it is projective in $\mathfrak{C}^{\text{opp}}$.

Much attention was given in Chapter I to the relation of projective modules to free modules. We now introduce the notion of a free object in an arbitrary category.

Definition. Let the category $\mathfrak{C}$ be equipped with an *underlying functor to sets*, that is, a functor $U: \mathfrak{C} \to \mathfrak{S}$ which is injective on morphisms, and let $Fr \dashv U$. Then, for any set S, $Fr(S)$ is called the *free object on S* (relative to U).

After the introduction to adjoint functors of Sections 7 and 8, the reader should have no difficulty in seeing that $Fr(S)$ has precisely the universal property we would demand of the free object on S. We will be concerned with two questions: (a) are free objects projective, (b) is every object the image of a free (or projective) object? We first note the following property of the category of sets.

Proposition 10.1. *In $\mathfrak{S}$ every object is both projective and injective.* □

We now prove

Proposition 10.2. *Let $F:\mathfrak{C}\to\mathfrak{D}$ and $F\dashv G$. If G maps epimorphisms to epimorphisms, then F maps projectives to projectives.*

Proof. Let P be a projective object of $\mathfrak{C}$ and consider the diagram in $\mathfrak{D}$,

$$\begin{array}{ccc} & & F(P) \\ & & \downarrow{\scriptstyle\varphi} \\ A & \xrightarrow{\ \varepsilon\ } & B \end{array}$$

Applying the adjugant, this gives rise to a diagram

$$\begin{array}{ccc} & & P \\ & & \downarrow{\scriptstyle\eta(\varphi)} \\ GA & \xrightarrow{\ G\varepsilon\ } & GB \end{array}$$

in $\mathfrak{C}$, where, by hypothesis, $G\varepsilon$ remains epimorphic. There thus exists $\psi':P\to GA$ in $\mathfrak{C}$ with $G\varepsilon\circ\psi'=\eta(\varphi)$, so that $\varepsilon\circ\psi=\varphi$, where $\eta(\psi)=\psi'$. □

Corollary 10.3. *If the underlying functor $U:\mathfrak{C}\to\mathfrak{S}$ sends epimorphisms to surjections then every free object in $\mathfrak{C}$ is projective.* □

This is the case, for example, for $\mathfrak{Ab}$, $\mathfrak{M}_A$, $\mathfrak{G}$; the hypothesis is false, however, for the category of integral domains, where, as the reader may show, the inclusion $\mathbb{Z}\subsetneqq\mathbb{Q}$ is an epimorphism (see Exercise 3.2).

We now proceed to the second question and show

Proposition 10.4. *Let $Fr\dashv U$, where $U:\mathfrak{C}\to\mathfrak{S}$ is the underlying functor. Then the counit $\delta:FrU(A)\to A$ is an epimorphism.*

Proof. Suppose $\alpha,\alpha':A\to B$ and $\alpha\circ\delta=\alpha'\circ\delta$. Applying the adjugant we find $U(\alpha)=U(\alpha')$. But U is injective on morphisms so $\alpha=\alpha'$. □

Thus every object admits a free presentation by means of the free object on its underlying set and this free presentation is a projective presentation if U sends epimorphisms to surjections.

Proposition 10.5. (i) *Every retract of a projective object is projective.*

(ii) *If U sends epimorphisms to surjections, then every projective object is a retract of a free (projective) object.*

Proof. (i) Given $P\underset{\sigma}{\overset{\varrho}{\rightleftarrows}}Q$, $\varrho\sigma=1$, P projective, and

$$\begin{array}{ccc} & & Q \\ & & \downarrow{\scriptstyle\varphi} \\ A & \xrightarrow{\ \varepsilon\ } & B, \end{array}$$

choose $\psi': P \to A$ so that $\varepsilon\psi' = \varphi\varrho$ and set $\psi = \psi'\sigma$. Then

$$\varepsilon\psi = \varepsilon\psi'\sigma = \varphi\varrho\sigma = \varphi .$$

(ii) Since δ is an epimorphism it follows that if A is projective there exists $\sigma: A \to FrU(A)$ with $\delta\sigma = 1$. Note that, even without the hypothesis on U, a projective P is a retract of $FrU(P)$; the force of the hypothesis is that then $FrU(P)$ is itself projective. □

Proposition 10.6. (i) *A coproduct of free objects is free.*
(ii) *A coproduct of projective objects is projective.*

Proof. (i) Since Fr has a right adjoint, it maps coproducts to coproducts. (Coproducts in $\mathfrak{S}$ are disjoint unions.)
(ii) Let $P = \coprod_i P_i$, P_i projective, and consider the diagram

$$\begin{array}{ccc} & & P \\ & & \downarrow{\scriptstyle\varphi} \\ A & \xrightarrow{\varepsilon} & B \end{array}$$

Then $\varphi = \langle\varphi_i\rangle$, $\varphi_i: P_i \to B$ and, for each i, we have $\psi_i: P_i \to A$ with $\varepsilon\psi_i = \varphi_i$. Then if $\psi = \langle\psi_i\rangle$, we have $\varepsilon\psi = \varphi$. Notice that, if the morphism sets of $\mathfrak{C}$ are non-empty then if P is projective so is each P_i by Proposition 10.5 (i). □

We shall have nothing to say here about *injective* objects beyond those remarks which simply follow by dualization.

Exercises:

10.1. Use Proposition I. 8.1 to prove that if Λ is free as an abelian group, then every free Λ-module is a free abelian group. (Of course, there are other proofs!).

10.2. Verify in detail that $Fr(S)$ has the universal property we would demand of the free object on S in the case $\mathfrak{C} = \mathfrak{G}$.

10.3. Deduce by a categorical argument that if $\mathfrak{C} = \mathfrak{G}$, then $Fr(S \cup T)$ is the free product of $Fr(S)$ and $Fr(T)$ if $S \cap T = \emptyset$.

10.4. Dualize Proposition 10.5.

10.5. Show that $\mathbb{Z} \subseteq \mathbb{Q}$ is an epimorphism (i) in the category of integral domains, (ii) in the category of commutative rings. Are there free objects in these categories which are not projective?

10.6. Let Λ be a ring not necessarily having a unity element. A (left) Λ-module is defined in the obvious way, simply suppressing the axiom $1a = a$. Show that Λ, as a (left) Λ-module, need not be free!