5. Two Exact Sequences

Here we shall deduce two exact sequences connecting Hom and Ext. We start with the following very useful lemma.

Lemma 5.1. *Let the following commutative diagram have exact rows.*

$$\begin{array}{ccccccc} & & A & \xrightarrow{\mu} & B & \xrightarrow{\varepsilon} & C & \longrightarrow 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & \\ 0 & \longrightarrow & A' & \xrightarrow{\mu'} & B' & \xrightarrow{\varepsilon'} & C' & \end{array}$$

Then there is a "connecting homomorphism" $\omega:\ker\gamma\to\operatorname{coker}\alpha$ *such that the following sequence is exact:*

$$\ker\alpha\xrightarrow{\mu_*}\ker\beta\xrightarrow{\varepsilon_*}\ker\gamma\xrightarrow{\omega}\operatorname{coker}\alpha\xrightarrow{\mu'_*}\operatorname{coker}\beta\xrightarrow{\varepsilon'_*}\operatorname{coker}\gamma\,. \quad (5.1)$$

If μ *is monomorphic, so is* μ_**: if* ε' *is epimorphic, so is* ε'_*.

Proof. It is very easy to see – and we leave the verification to the reader – that the final sentence holds and that we have exact sequences

$$\ker\alpha\xrightarrow{\mu_*}\ker\beta\xrightarrow{\varepsilon_*}\ker\gamma\,,$$

$$\operatorname{coker}\alpha\xrightarrow{\mu'_*}\operatorname{coker}\beta\xrightarrow{\varepsilon'_*}\operatorname{coker}\gamma\,.$$

It therefore remains to show that there exists a homomorphism $\omega:\ker\gamma\to\operatorname{coker}\alpha$ "connecting" these two sequences. In fact, ω is defined as follows.

Let $c\in\ker\gamma$, choose $b\in B$ with $\varepsilon b=c$. Since $\varepsilon'\beta b=\gamma\varepsilon b=\gamma c=0$ there exists $a'\in A'$ with $\beta b=\mu' a'$. Define $\omega(c)=[a']$, the coset of a' in $\operatorname{coker}\alpha$.

We show that ω is well defined, that is, that $\omega(c)$ is independent of the choice of b. Indeed, let $\bar b\in B$ with $\varepsilon\bar b=c$, then $\bar b=b+\mu a$ and

$$\beta(b+\mu a)=\beta b+\mu'\alpha a\,.$$

Hence $\bar a'=a'+\alpha a$, thus $[\bar a']=[a']$. Clearly ω is a homomorphism.

Next we show exactness at $\ker\gamma$. If $c\in\ker\gamma$ is of the form εb for $b\in\ker\beta$, then $0=\beta b=\mu' a'$, hence $a'=0$ and $\omega(c)=0$. Conversely, let $c\in\ker\gamma$ with $\omega(c)=0$. Then $c=\varepsilon b$, $\beta b=\mu' a'$ and there exists $a\in A$ with $\alpha a=a'$. Consider $\bar b=b-\mu a$. Clearly $\varepsilon\bar b=c$, but

$$\beta\bar b=\beta b-\beta\mu a=\beta b-\mu' a'=0\,,$$

hence $c\in\ker\gamma$ is of the form $\varepsilon\bar b$ with $\bar b\in\ker\beta$.

Finally we prove exactness at $\operatorname{coker}\alpha'$. Let $\omega(c)=[a']\in\operatorname{coker}\alpha$. Thus $c=\varepsilon b$, $\beta b=\mu' a'$, and $\mu'_*[a']=[\mu' a']=[\beta b]=0$. Conversely, let $[a']\in\operatorname{coker}\alpha$ with $\mu'_*[a']=0$. Then $\mu' a'=\beta b$ for some $b\in B$ and $c=\varepsilon b\in\ker\gamma$. Thus $[a']=\omega(c)$. □

1.4. Show that Z_n, B_n depend functorially on the complex.

1.5. Let C be a free abelian chain complex with $C_n = 0$, $n < 0, n > N$. Let ϱ_n be the rank of C_n and let p_n be the rank of $H_n(C)$. Show that

$$\sum_{n=0}^{N} (-1)^n \varrho_n = \sum_{n=0}^{N} (-1)^n p_n .$$

1.6. Given a chain complex C of right Λ-modules, a left Λ-module A and a right Λ-module B, suggest definitions for the chain complex $C \otimes_\Lambda A$, and the cochain complex $\mathrm{Hom}_\Lambda(C. B)$.

2. The Long Exact (Co)Homology Sequence

We have already remarked that the category of (co)chain complexes is abelian. Consequently we can speak of short exact sequences of (co)chain complexes. It is clear that the sequence $A \overset{\varphi}{\rightarrowtail} B \overset{\psi}{\twoheadrightarrow} C$ of complexes is short exact if and only if $0 \to A_n \xrightarrow{\varphi_n} B_n \xrightarrow{\psi_n} C_n \to 0$ is exact for all $n \in \mathbb{Z}$.

Theorem 2.1. *Given a short exact sequence $A \overset{\varphi}{\rightarrowtail} B \overset{\psi}{\twoheadrightarrow} C$ of chain complexes (cochain complexes) there exists a morphism of degree -1 (degree $+1$) of graded modules $\omega: H(C) \to H(A)$ such that the triangle*

$$\begin{array}{ccc} H(A) & \xrightarrow{\quad \varphi_* \quad} & H(B) \\ & {}_{\omega}\nwarrow \quad \swarrow_{\psi_*} & \\ & H(C) & \end{array}$$

is exact. (We call ω the connecting homomorphism.)

Explicitly the theorem claims that, in the case of chain complexes, the sequence

$$\cdots \xrightarrow{\omega_{n+1}} H_n(A) \xrightarrow{\varphi_*} H_n(B) \xrightarrow{\psi_*} H_n(C) \xrightarrow{\omega_n} H_{n-1}(A) \to \cdots \tag{2.1}$$

and, in the case of cochain complexes, the sequence

$$\cdots \xrightarrow{\omega^{n-1}} H^n(A) \xrightarrow{\varphi_*} H^n(B) \xrightarrow{\psi_*} H^n(C) \xrightarrow{\omega^n} H^{n+1}(A) \to \cdots \tag{2.2}$$

is exact.

We first prove the following lemma.

Lemma 2.2. $\partial_n: C_n \to C_{n-1}$ *induces* $\tilde{\partial}_n: \mathrm{coker}\,\partial_{n+1} \to \ker \partial_{n-1}$ *with* $\ker \tilde{\partial}_n = H_n(C)$ *and* $\mathrm{coker}\,\tilde{\partial}_n = H_{n-1}(C)$.

Proof. Since $\mathrm{im}\,\partial_{n+1} \subseteq \ker \partial_n$ and $\mathrm{im}\,\partial_n \subseteq \ker \partial_{n-1}$ the differential ∂_n induces a map $\tilde{\partial}_n$ as follows:

$$\mathrm{coker}\,\partial_{n+1} = C_n / \mathrm{im}\,\partial_{n+1} \twoheadrightarrow C_n / \ker \partial_n \cong \mathrm{im}\,\partial_n \subseteq \ker \partial_{n-1} .$$

One easily computes $\ker \tilde{\partial}_n = \ker \partial_n / \mathrm{im}\,\partial_{n+1} = H_n(C)$ and

$$\mathrm{coker}\ \tilde{\partial}_n = \ker \partial_{n-1} / \mathrm{im}\,\partial_n = H_{n-1}(C) . \quad \square$$

Proof of Theorem 2.1. We give the proof for chain complexes only, the proof for cochain complexes being analogous. We first look at the diagram

$$\begin{array}{ccccccc}
0 \longrightarrow \ker\partial_n & \longrightarrow & \ker\partial_n & \longrightarrow & \ker\partial_n & & \\
\vdots & & \vdots & & \vdots & & \\
A_n & \xrightarrow{\varphi_n} & B_n & \xrightarrow{\psi_n} & C_n & & \\
\downarrow{\scriptstyle \partial_n} & & \downarrow{\scriptstyle \partial_n} & & \downarrow{\scriptstyle \partial_n} & & \\
A_{n-1} & \xrightarrow{\varphi_{n-1}} & B_{n-1} & \xrightarrow{\psi_{n-1}} & C_{n-1} & & \\
\vdots & & \vdots & & \vdots & & \\
\operatorname{coker}\partial_n & \longrightarrow & \operatorname{coker}\partial_n & \longrightarrow & \operatorname{coker}\partial_n & \longrightarrow & 0
\end{array}$$

By Lemma III. 5.1 the sequence at the top and the sequence at the bottom are exact. Thus by Lemma 2.2 we obtain the diagram

$$\begin{array}{ccccccc}
H_n(A) & \longrightarrow & H_n(B) & \longrightarrow & H_n(C) & & \\
\vdots & & \vdots & & \vdots & & \\
\operatorname{coker}\partial_{n+1} & \longrightarrow & \operatorname{coker}\partial_{n+1} & \longrightarrow & \operatorname{coker}\partial_{n+1} & \longrightarrow & 0 \\
\downarrow{\scriptstyle \tilde\partial_n} & & \downarrow{\scriptstyle \tilde\partial_n} & & \downarrow{\scriptstyle \tilde\partial_n} & & \\
0 \longrightarrow \ker\partial_{n-1} & \longrightarrow & \ker\partial_{n-1} & \longrightarrow & \ker\partial_{n-1} & & \\
\vdots & & \vdots & & \vdots & & \\
H_{n-1}(A) & \longrightarrow & H_{n-1}(B) & \longrightarrow & H_{n-1}(C) & &
\end{array}$$

Applying Lemma III. 5.1 again we deduce the existence of

$$\omega_n : H_n(C) \longrightarrow H_{n-1}(A)$$

such that the sequence (2.1) is exact. □

If we recall the explicit definition of ω_n, then it is seen to be equivalent to the following procedure. Let $c \in C_n$ be a representative cycle of the homology class $[c] \in H_n(C)$. Choose $b \in B_n$ with $\psi(b) = c$. Since (suppressing the subscripts) $\psi\partial b = \partial\psi b = \partial c = 0$ there exists $a \in A_{n-1}$ with $\varphi a = \partial b$. Then $\varphi\partial a = \partial\varphi a = \partial\partial b = 0$. Hence a is a cycle in $Z_{n-1}(A)$ and therefore determines an element $[a] \in H_{n-1}(A)$. The map ω_n is defined by $\omega_n[c] = [a]$.

We remark that the naturality of the ker-coker sequence of Lemma III. 5.1 immediately implies the naturality of sequences (2.1) and (2.2). If we are given a commutative diagram of chain complexes

$$\begin{array}{ccccc}
A & \rightarrowtail & B & \twoheadrightarrow & C \\
\downarrow & & \downarrow & & \downarrow \\
A' & \rightarrowtail & B' & \twoheadrightarrow & C'
\end{array}$$