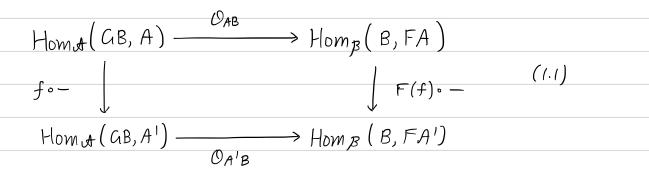
In this lecture we (finally) discuss <u>adjunctions</u>, which have secretly been playing an important wile since the beginning of the course.

<u>Def</u> Let $F: \mathcal{A} \longrightarrow \mathcal{B}$ and $G: \mathcal{B} \longrightarrow \mathcal{A}$ be functors. An <u>adjunction</u> with G on the left and F on the right is a family of bijections

 \mathcal{O}_{AB} : Hom_A (GB, A) $\xrightarrow{\simeq}$ Hom_B (B, FA), A \in ob(A), B \in ob(B)

which is natural in A, B by which we mean that for every morphism $f: A \longrightarrow A'$ in A the diagram



commutes, and for every morphism $g: B \longrightarrow B'$ in β the diagram

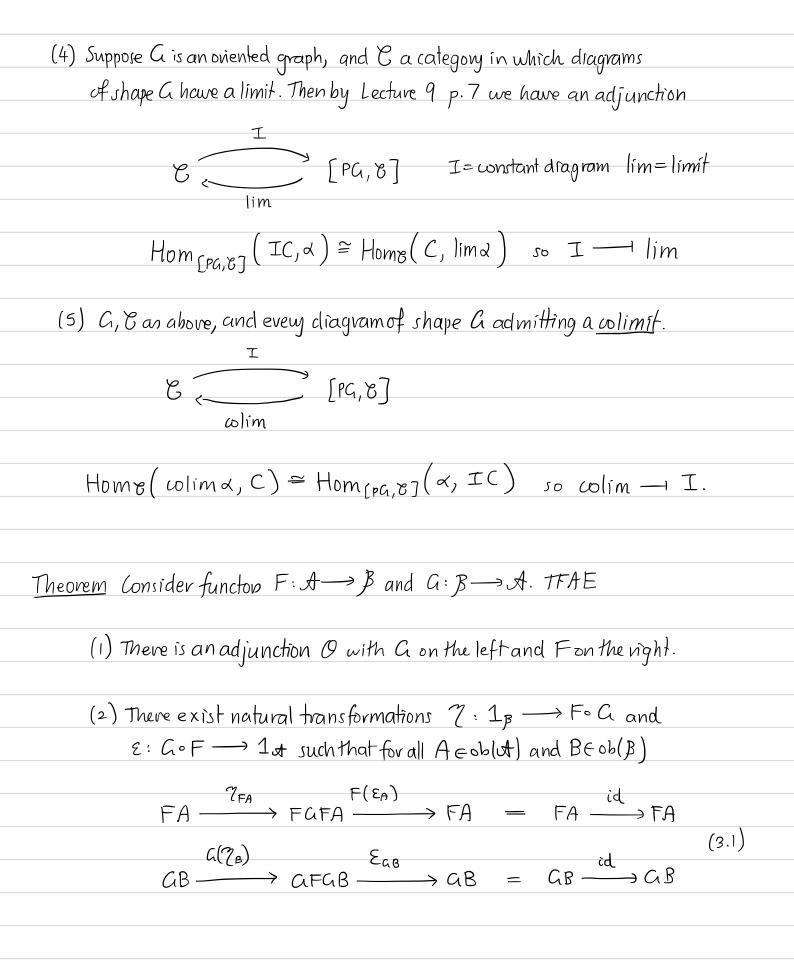
$$\begin{array}{ccc} \text{Hom}_{\mathcal{A}}(GB,A) & \xrightarrow{\mathcal{O}_{AB}} & \text{Hom}_{\mathcal{B}}(B,FA) \\ \hline & & & & \\ \hline & & & \\ -\circ Gg & & & \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ -\circ g & & \\ \hline & & & \\ \hline & & & \\ -\circ g & & \\ \hline & & & \\ \hline & & & \\ -\circ g & & \\ \hline & & & \\ \hline & & & \\ -\circ g & & \\ \hline & & & \\ \hline & & & \\ -\circ g & & \\ \hline & & & \\ \hline & & & \\ -\circ g & & \\ \hline & & & \\ \hline & & & \\ -\circ g & & \\ \hline & & & \\ \hline & & & \\ -\circ g & & \\ \hline & & & \\ \hline & & & \\ -\circ g & & \\ \hline & & & \\ \hline & & & \\ -\circ g & & \\ \hline & & & \\ \hline & & & \\ -\circ g & & \\ \hline & & & \\ \hline & & & \\ -\circ g & & \\ \hline & & & \\ \hline & & & \\ -\circ g & & \\ \hline & & & \\ \hline & & & \\ -\circ g & & \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ -\circ g & & \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ -\circ g & & \\ \hline & & & \\ \hline \end{array} \\ \hline & & & \\ \hline \hline & & & \\ \hline \end{array} \end{array}$$

commutes.

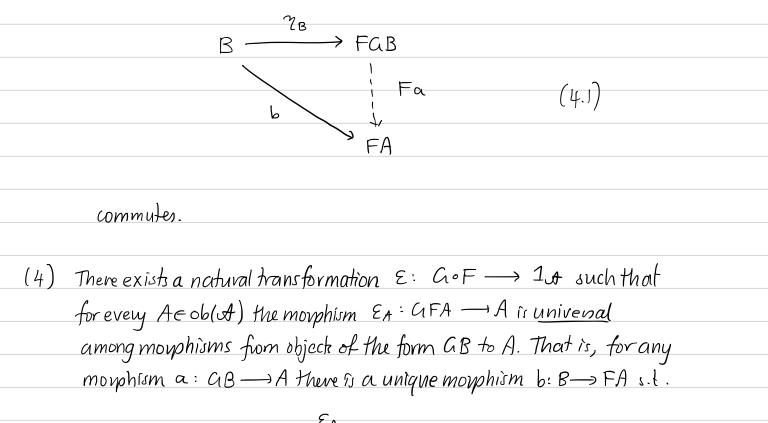
If such an adjunction exists we say G is left adjoint to F, that F is night adjoint to G, and write G - I F. We call (F,G) an adjoint pair.

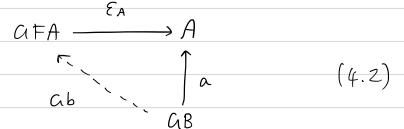
Note As we will show, there is essentially at most one adjunction (i.e. family of O's) between a pair F, G and the right (resp. left) adjoint of a functor is essentially unique if it exists.

$$\underbrace{Examples}_{F} (1) (Lecture 2, p.10) There is an adjoint pair
Set \swarrow Ab $F = forget ful, V = free ab.grp.$
 $Hom_{Ab}(VS, B) \cong Hom_{set}(S, FB)$ so $V \longrightarrow F$
(2) (Lecture 2, Ex11) There is an adjoint pair
 $Mon \bigoplus_{F} Rng$ $F = forget ful, V = free ving on a monoid$
 $Hom_{eng}(VG, R) \cong Hom_{Hom}(G, FR)$ so $V \longrightarrow F$
(3) (Lecture 7, p.2) There is an adjoint pair
 $Guaph \bigoplus_{U} Gat$ $U = underlying, P = path cat.$
 $Hom_{cat}(PG, S) \cong Hom_{Gleph}(G_1 \cup S)$ so $P \longrightarrow U$
 $Upshof : free is left adjoint to forget ful !$$$



(3) There exists a natural transformation ?: 1_B → F°G such that for every BEOD(B) the morphism ?_B: B → FGB is <u>universal</u> among morphisms from B to objects of the form FA. That is, for any morphism b: B → FA there is a unique morphism a: GB → A such that





commutes.

<u>Proof</u> We prove $(1) \Leftrightarrow (3)$ and leave $(2) \Leftrightarrow (1) \Leftrightarrow (4)$ as an exercise. First, $(1) \Rightarrow (3)$. Suppose we have natural bijections

 \mathcal{O}_{AB} : Hom_A (GB, A) $\xrightarrow{\cong}$ Hom_B (B, FA),

 \oplus

and observe that taking
$$A=GB$$
 gives
 $\mathcal{O}_{GB,B}$: Hom_A $(GB,GB) \xrightarrow{\sim}$ Hom_B (B, FGB) ,
and we define
 $\mathcal{T}_{B} := \mathcal{O}_{GB,9}(I_{AB}) : B \longrightarrow FGB$.
We need to check this is (a) natural and (b) universal. For naturality,
suppose $f: B \rightarrow B'$ is given. By naturality of \mathcal{O}_{r} both squares in
Hom_ot $(GB, GB) \xrightarrow{\mathcal{O}_{GB,S}} Hom_{p}(B, FGB)$
 $af \circ - \downarrow \qquad fra(f) \circ -$
 $Hom_{ot}(GB, GB') \xrightarrow{\mathcal{O}_{GB,B}} Hom_{p}(B, FGB')$ (f, f)
 $-\circ Gf \qquad \qquad f \circ - f$
 $Hom_{ot}(GB', GB') \xrightarrow{\mathcal{O}_{GB,S}} Hom_{p}(B', FGB')$
 $commute. The top one says$
 $FG(f) \circ \mathcal{O}_{GB,B}(I_{GB}) = \mathcal{O}_{GB',B}(Gf)$
 $i.e. FG(f) \circ \mathcal{T}_{B} = \mathcal{O}_{GB',B}(Gf)$

while the bottom one says

$$O_{GB',B}(l_{AB} \circ GF) = O_{GB',B'}(l_{AB'}) \circ f$$

$$I \cdot e \cdot G_{GB',B}(GF) = \gamma_{B'} \circ f$$

$$We conclude \quad FG(f) \circ \gamma_{B} = \gamma_{B'} \circ f \quad Jo \quad \gamma \text{ is a natural transformation}$$

$$I_{p} \rightarrow F \circ G \cdot I + \text{ remains to prove the universal property, but we will}$$

$$derive \text{ this easily from the following observation: given any } q \cdot GB \rightarrow A$$

$$(\text{that is, Jomething to which } O_{A,B} \text{ may be applied}) \text{ we may apply}$$

$$naturality \text{ of } O \text{ to deduce commutativity of}$$

$$Hom_{A}(GB, GB) \xrightarrow{O_{GB,B}} Hom_{B}(B, FGB)$$

$$a \circ - \int \int F(a) \circ - (6 \cdot 1)$$

$$Hom_{A}(GB, A) \xrightarrow{O} Hom_{B}(B, FA).$$

But this says in particular

$$\mathcal{O}_{AB,A}(a) = \mathcal{O}_{AB,A}(a \circ |_{AB})$$

= $F(a) \circ \mathcal{O}_{AB,B}(|_{AB})$ (6.2)
= $F(a) \circ \mathcal{Y}_{B}$.

Thus O is determined by 2, just as 2 is determined by O. To see any this implies (3), let $b: B \rightarrow FA$ be given. Considering (6.1) and the fact that O is a bijection, there must be a unique $q: GB \rightarrow A$ with $O_{AB,A}(q) = b$. But this is the same (by (6.2)) as to say there is a unique a with $b = F(q) \circ 2c$, which is what we wanted to show. Hence $(1) \Rightarrow (3)$.

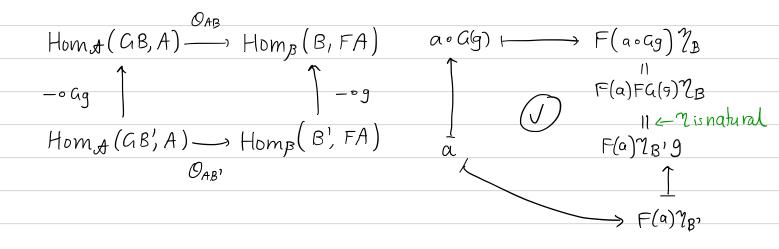
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For (3) \Rightarrow (1) suppose we have $\gamma: 1_p \rightarrow F \circ G$ with the universal property and define the function

$$\begin{array}{c} \mathcal{O}_{A,B} : \operatorname{Hom}_{\mathcal{A}}(GB,A) \longrightarrow \operatorname{Hom}_{\mathcal{B}}(B,FA) \\ (7.1) \\ \mathcal{O}_{A,B}(a) = F(a) \circ \mathcal{V}_{B} \\ \hline \mathcal{V}_{B} & \xrightarrow{\mathcal{V}_{B}} FaB \xrightarrow{Fa} FA \end{array}$$

The universal properly of
$$\gamma$$
 says precisely that $\mathcal{O}_{A,B}$ is a bijection. It remains
to check naturality of \mathcal{O} . Given $f: A \longrightarrow A'$ and $g: B \longrightarrow B'$ we have to check
commutativity of



So this shows $(3) \Rightarrow (1)$.

 \bigcirc

Expl complete the proof of the theorem by proving $(1) \iff (2)$ and $(1) \iff (4)$.

<u>Def</u>^{*} In an adjunction the natural transformation $7:1_3 \rightarrow F \circ G$ is called the unit and $\Sigma: G \circ F \longrightarrow 1_{A}$ the <u>counit</u>.

- Ezz Write down the unit and counit transformations in the examples (1)-(5) on p-(2),(3). Here you will use the explicit O's defined in earlier lectures.
 - Ex 3 Rove that if a functor F: A→B has a right (vesp-left) adjoint, it is unique up to isomorphism (and in fact up to <u>unique</u> isomorphism if we add a natural constraint involving either the unit or counit. What is the night constraint?).

Ex 4 Given a pair of function F, G as above, to what extent is an adjunction O between them unique (if it exists)?