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# Tutorial 8 : Integration

The following is taken from T. Tao's "Analysis" Vol. 1 Ch. 11. A subset $I \subseteq \mathbb{R}$ is an interval if there exist $a \leq b$ with $I$ equal to one of the following sets:

except we do not allow empty intervals.

$$[a,b],\ (a,b],\ [a,b),\ (a,b).$$

In all cases we define the length of $I$ to be $|I| := b-a$ (possibly zero). A partition of an interval $I$ will mean a finite set $P$ whose elements are pairwise disjoint intervals contained in $I$, whose union is all of $I$.

Example $P_1 = \{[0,1/3),[1/3,1]\}$, $P_2 = \{[0,1/2),[1/2,1]\}$ are partitions of $[0,1]$.

Given partitions $P_1, P_2$ of $I$ we write $P_1 \leq P_2$ (not $\subseteq$) if for every $x \in P_1$ there exists $y \in P_2$ with $x \subseteq y$. This is a partial order on the set of partitions of $I$, and moreover given partitions $P_1, P_2$

$$P_1 \wedge P_2 := \{ J \cap K \mid J \in P_1, K \in P_2 \text{ and } J \cap K \neq \emptyset \}$$

is another partition of $I$ with the property that $P_1 \wedge P_2 \leq P_i$ for $i \in \{1,2\}$ and if $Q$ is another partition with $Q \leq P_1$, $Q \leq P_2$ then $Q \leq P_1 \wedge P_2$

$\boxed{Q1}$ Prove $\leq$ is a partial order on the set of partitions of $I$.

$\boxed{Q2}$ Given a partition $P$ of $I$ prove that $|I| = \sum_{x \in P} |x|$. (Hint: argue the statement for all pairs $(I, P)$ by induction on the size of $P$).

<u>Def$^n$</u> Given an interval $I$ with partition $\mathcal{P}$, a function $f: I \longrightarrow \mathbb{R}$ is <u>piecewise constant with respect to $\mathcal{P}$</u> if for all $J \in \mathcal{P}$, the function $f|_J : J \longrightarrow \mathbb{R}$ is a constant function. A function $f: I \longrightarrow \mathbb{R}$ is <u>piecewise constant</u> if it is piecewise constant with respect to <u>some</u> partition $\mathcal{P}$ of $I$.

$\boxed{Q3}$ Prove if $f: I \longrightarrow \mathbb{R}$ is piecewise constant with respect to partitions $\mathcal{P}_1, \mathcal{P}_2$ then

$$\sum_{J \in \mathcal{P}_1} a_J |J| = \sum_{K \in \mathcal{P}_2} b_K |K| \qquad (*)$$

where for $J \in \mathcal{P}_1$ we have $f|_J \equiv a_J$ and for $K \in \mathcal{P}_2$, $f|_K \equiv b_K$ for constants $a_J, b_K \in \mathbb{R}$. This common value $(*)$ which is independent of the partition we denote by $\text{p.c.}\int_I f$. (Hint: use $\mathcal{P}_1 \wedge \mathcal{P}_2$ to reduce to the case $\mathcal{P}_1 \subseteq \mathcal{P}_2$).

<u>Def$^n$</u> Let $f: I \longrightarrow \mathbb{R}$ be a bounded function on an interval $I$. The <u>upper Riemann integral</u> is the real number

$$\overline{\int_I} f := \inf\Big\{ \text{p.c.}\int_I g \;\Big|\; g \text{ is piecewise constant on } I \text{ and for all } x \in I, \text{ we have } g(x) \geq f(x) \Big\}$$

while the <u>lower Riemann integral</u> is the real number

$$\underline{\int_I} f := \sup\Big\{ \text{p.c.}\int_I g \;\Big|\; g \text{ is piecewise constant on } I \text{ and for all } x \in I, \text{ we have } g(x) \leq f(x) \Big\}.$$

If $\overline{\int_I} f = \underline{\int_I} f$ we say $f$ is <u>Riemann integrable</u> and define

$$\int_I f := \overline{\int_I} f = \underline{\int_I} f.$$

Theorem Any continuous function $f : [a,b] \to \mathbb{R}$ is Riemann integrable.

Q4 Prove the theorem

[Long hint: aim to show that for any $\varepsilon > 0$ (with $I = [a,b]$)

$$\overline{\int_I} f - \underline{\int_I} f \leqslant \varepsilon(b-a).$$

To do this, produce a partition $J_1, \dots, J_n$ of $[a,b]$ (whose elements and length both depend on $\varepsilon$) and constants $p_1, \dots, p_n, q_1, \dots, q_n$ such that

$$\overline{\int_I} f \leq \sum_{i=1}^{n} p_i |J_i|, \qquad \underline{\int_I} f \geqslant \sum_{i=1}^{n} q_i |J_i|$$

so that $\overline{\int_I} f - \underline{\int_I} f \leqslant \sum_{i=1}^{n} (p_i - q_i)|J_i|$.

Then use that a continuous function $f$ on $[a,b]$ is <u>uniformly</u> continuous, i.e. $\forall \varepsilon > 0 \, \exists \delta > 0 \, \forall x, y \in [a,b] (\, |x-y| < \delta \Rightarrow |fx - fy| < \varepsilon)$. This is an easy consequence of compactness.]

Q5 Prove that if $f : [a,b] \to \mathbb{R}$ is continuous and $a < c < b$ then

$$\int_{[a,b]} f = \int_{[a,c]} f + \int_{[c,b]} f.$$

<u>Def$^n$</u> A continuous function $f : X \to \mathbb{R}$ is <u>compactly supported</u> if there is $K \subseteq X$ compact such that if $x \notin K$ then $f(x) = 0$.

Q6 Prove that if $f : \mathbb{R} \to \mathbb{R}$ is compactly supported, and $[a,b]$ is such that $f$ is zero outside $[a,b]$, then $\int_{[a,b]} f$ is independent of $a, b$, and we call this invariant quantity the <u>Riemann integral of $f$</u>.