We begin with the commutative diagram \((A = [F^7]^7)\)

\[
\begin{array}{ccc}
\mathbb{R}^n & \xrightarrow{MA} & \mathbb{R}^m \\
\uparrow c_\beta & = & \uparrow c_\psi \\
V & \xrightarrow{F} & W
\end{array}
\]

Applying \((-)^*\) to each of the linear maps and using \((fog)^* = g^* f^*\) we see that the bottom square in the following diagram commutes:

\[
\begin{array}{ccc}
\mathbb{R}^n & \xleftarrow{MA^*} & \mathbb{R}^m \\
\downarrow (4.2) & & \downarrow (4.2) \\
\mathbb{R}^n^* & \xleftarrow{M_A^*} & \mathbb{R}^m^* \\
\downarrow (C_\beta^*)^{-1} & = & \uparrow (C_\psi^*)^{-1} \\
V^* & \xleftarrow{F^*} & W^*
\end{array}
\]

Since the vertical maps are by def \(N C_\beta^*\) and \(C_\psi^*\) we need only show the diagram marked (?) commutes. But \(k^n \rightarrow k^n \rightarrow (k^n)^*\) sends

\[
e_j \mapsto A^T e_j = \sum_{i=1}^n (A^T)_{ij} e_i = \sum_{i=1}^n A_{ji} e_i \mapsto \sum_{i=1}^n A_{ji} e_i^*
\]

while \(k^m \rightarrow (k^m)^* \xrightarrow{M_A^*} (k^n)^*\) sends \(e_j \mapsto e_j^* \mapsto e_j^* o MA\) and \(e_j^* o MA = \sum_e A_{ji} e_i^*\) since they both agree on a basis element \(e_i\), since

\[
(e_j^* o MA)(e_i) = e_j^* (\sum_{i=1}^m A_{xi} e_i) = A_{ji} e_i.
\]
First we prove

$\left[ v \right]^T_B Q \left[ w \right]_B = B(v,w)$.

The left hand side and right hand side are both linear in $v, w$, so it is equivalent to check

$\left[ v_i \right]^T_B Q \left[ v_j \right]_B = B(v_i,v_j)$

for all $i,j$. But this is true by def.4, since $Q_{ij} = ([B]^{\beta^*}_B)^T j = B(v_j)(v_i)$.

F is adjoint to $G \iff p_B \circ F = A^* \circ p_B$

$\iff [p_B \circ F]^{\beta^*}_B = [A^* \circ p_B]^{\beta^*}_B$


$\iff Q [F]^{\beta}_B = ([A]^{\beta}_B)^T Q$

$\iff Q [F]^{\beta}_B Q^{-1} = ([A]^{\beta}_B)^T Q$