Tutorial 7 : Dual spaces

Dual vector spaces and the duals of linear maps will be central objects of study in the remainder of the course, and this tutorial prepares the ground with some of the basics. Throughout k is a field and vector spaces are over k, not necessarily finite-dimensional.

Def^n Given vector spaces V, W the set

$$\mathrm{Lin}(V,W) := \{ f : V \to W \mid f \text{ is linear} \}$$

is a vector space with the "pointwise" operations defined for $f, g \in \mathrm{Lin}(V,W)$ and $\lambda \in \mathbb{R}$ by

- $(f+g)(v) = f(v) + g(v)$
- $(-f)(v) = -f(v)$
- $0(v) = 0_W$. $\qquad$ (i.e. $0 \in \mathrm{Lin}(V,W)$)
- $(\lambda f)(v) = \lambda \cdot f(v)$.

Def^n The dual space V^* of a vector space V is $V^* := \mathrm{Lin}(V,k)$. We often refer to vectors $\mu \in V^*$ as functionals.

Clearly if U, V, W are vector spaces and $f: U \to V$, $g : V \to W$ are linear then the functions given below are also linear:

$$\mathrm{Lin}(U,V) \xrightarrow{g \circ (-)} \mathrm{Lin}(U,W) \qquad h \mapsto g \circ h$$

$$\mathrm{Lin}(V,W) \xrightarrow{(-) \circ f} \mathrm{Lin}(U,W) \qquad h \mapsto h \circ f$$

<u>Def^n</u> If $f: U \to V$ is linear then the linear transformation

$$V^* = \mathrm{Lin}(V, k) \longrightarrow \mathrm{Lin}(U, k) = U^*$$
$$h \longmapsto h \circ f$$

is denoted $f^*: V^* \to U^*$.

$\boxed{Q1}$ Prove that $(1_U)^* = 1_{U^*}$ and if $f: U \to V$, $g: V \to W$ are linear then $f^* \circ g^* = (g \circ f)^*$ as linear maps $W^* \to U^*$. We say the dual is a <u>functorial</u> operation. Conclude that if f is an isomorphism then so is f^*.

$\boxed{Q2}$ Prove that $(f+g)^* = f^* + g^*$ and $(\lambda f)^* = \lambda \cdot f^*$ so that taking the dual is a linear map $(-)^*: \mathrm{Lin}(V, W) \longrightarrow \mathrm{Lin}(W^*, V^*)$.

Given $A \in M_{m,n}(k)$ let $M_A: k^n \to k^m$ be $M_A(x) = Ax$. Then there is an isomorphism of vector spaces

$$M_{m,n}(k) \xrightarrow{\cong} \mathcal{L}(k^n, k^m)$$
$$A \longmapsto M_A$$

Suppose that V is <u>finite-dimensional</u> and that $\beta = (v_1, \ldots, v_n)$ is a sequence of vectors in V. Then β is an ordered basis <u>if and only if</u>

$$c_\beta: k^n \longrightarrow V, \qquad c_\beta(\lambda_1, \ldots, \lambda_n) = \sum_{i=1}^n \lambda_i v_i$$

is an isomorphism of vector spaces (it is always a well-defined linear map).

If W is another finite-dimensional vector space with ordered basis $\mathcal{C} = (w_1, \ldots, w_m)$ then given a linear transformation $F : V \to W$ there is a unique matrix A such that the following diagram commutes:

$$\begin{tikzcd}
k^n \arrow[r, "M_A"] \arrow[d, "c_\beta"', "\cong"] & k^m \arrow[d, "c_{\mathcal{C}}", "\cong"'] \\
V \arrow[r, "F"'] & W
\end{tikzcd} \qquad (*)$$

We call A the matrix of F with respect to $\beta, \mathcal{C}$. Some common notation:

- Given $v \in V$ we write $[v]_\beta := c_\beta^{-1}(v)$ (similarly $[w]_{\mathcal{C}}$).
- We write $[F]_\beta^{\mathcal{C}}$ for A. We may calculate from $(*)$ that the ith column of this matrix is

$$\begin{aligned}
(c_{\mathcal{C}}^{-1} \circ F \circ c_\beta)(e_i) &= (c_{\mathcal{C}}^{-1} \circ F)(v_i) \\
&= c_{\mathcal{C}}^{-1}(F(v_i)) \\
&= [F(v_i)]_{\mathcal{C}}.
\end{aligned}$$

$\boxed{\text{Q3}}$ Use the diagram $(*)$ to show that if $G : W \to U$ is another linear transformation and $\mathcal{D}$ is an ordered basis for U then

$$[G]_{\mathcal{C}}^{\mathcal{D}} \circ [F]_\beta^{\mathcal{C}} = [G \circ F]_\beta^{\mathcal{D}}.$$

Next we observe that a familiar matrix operation, the transpose, is "really" about dual spaces, in the following sense.

Definition If V is a vectorspace, a function $B : V \times V \longrightarrow \mathbb{R}$ is bilinear if

$$\begin{aligned}
&\text{(i)} \quad B(x+x', y) = B(x,y) + B(x',y) && \forall x, x', y \in V \\
&\text{(ii)} \quad B(x, y+y') = B(x,y) + B(x,y') && \forall x, y, y' \in V \\
&\text{(iii)} \quad B(\lambda x, y) = B(x, \lambda y) = \lambda B(x,y) && \forall x, y \in V \ \ \forall \lambda \in \mathbb{R}.
\end{aligned}$$

We say B is symmetric if $B(x,y) = B(y,x)$ for all $x, y \in V$.

$\boxed{Q4}$ Prove that there is a bijection between bilinear forms on V and linear maps $V \longrightarrow V^*$ where V^* denotes the dual space, where the linear map f_B associated to B is $f_B(x)(y) = B(x,y)$.

$$\begin{array}{ccc}
\left\{ \begin{array}{c} \text{bilinear pairings} \\ V \times V \longrightarrow k \end{array} \right\} & \xleftrightarrow{1:1} & \left\{ \begin{array}{c} \text{linear maps} \\ V \longrightarrow V^* \end{array} \right\} \\
B & \longmapsto & \{ v \mapsto B(v, -) \}
\end{array}$$

$\boxed{Q5}$ Suppose V is finite-dimensional. A symmetric bilinear form B on V is called nondegenerate if f_B is an isomorphism $V \xrightarrow{\cong} V^*$. Write down the definition of nondegeneracy purely in terms of pairs (x,y) for which $B(x,y) = 0$.

Example The pairing $B : k^n \times k^n \longrightarrow k$, $B(\underline{a}, \underline{b}) = \sum_{i=1}^n a_i b_i$ is nondegenerate, and the associated isomorphism is

$$\begin{array}{ccc}
k^n & \xrightarrow{\cong} & (k^n)^*, \\
e_i & \longmapsto & e_i^*
\end{array} \qquad (**)$$

where $e_i^*(\underline{a}) = a_i$. In particular $(e_1^*, \ldots, e_n^*)$ must be a basis for $(k^n)^*$, called the dual basis.

<u>Def</u>

Def^n Let V be a finite-dimensional vector space with ordered basis $\beta = (v_1, \ldots, v_n)$. The <u>dual basis</u> is $\beta^* = (v_1^*, \ldots, v_n^*)$ for V^* is

$$v_i^*(u) = e_i^*([u]_\beta)$$

which reads off the coefficient of v_i in the expansion of u w.r.t. β.

$\boxed{Q6}$ Prove β^* is a basis for V^*.

$\boxed{Q7}$ Suppose $V, W, \beta, \gamma, F, A$ are as in $(*)$ on p. 3. Prove that

$$\begin{tikzcd}
k^n \arrow[d, "C_{\beta^*}"'] & k^m \arrow[l, "M_{A^T}"'] \arrow[d, "C_{\gamma^*}"] \\
V^* & W^* \arrow[l, "F^*"]
\end{tikzcd}$$

commutes, and hence $[F^*]_{\gamma^*}^{\beta^*} = \left([F]_\beta^\gamma\right)^T$, so that taking the transpose <u>means</u> taking the dual of a linear transformation.

Solutions

Q6 Note that v_i^* is the linear map

$$V \xrightarrow{c_\beta^{-1}} k^n \xrightarrow{e_i^*} k$$

so that there is a well-defined linear map $c_{\beta^*} : k^n \longrightarrow V^*$ sending e_i to v_i^*, and we just need to show this is an isomorphism. But this follows from commutativity of

$$\begin{tikzcd}
k^n \arrow[rr, "c_{\beta^*}"] \arrow[dr, "\cong", "(**)"'] & & V^* \arrow[dl, "\cong"', "(c_\beta)^*"] \\
& (k^n)^* &
\end{tikzcd}$$

which we can verify on the basis $e_1, \dots, e_n$ by obsewing that

$$\begin{aligned}
(c_\beta)^* c_{\beta^*}(e_i)(e_j) &= c_\beta^*(v_i^*)(e_j) \\
&= [v_i^* \circ c_\beta](e_j) \\
&= [e_i^* \circ c_\beta^{-1} \circ c_\beta](e_j) \\
&= e_i^*(e_j) = \delta_{ij}
\end{aligned}$$

Hence $(c_\beta)^* c_{\beta^*}(e_i) = e_i^*$, as claimed.

$\boxed{Q7}$ To check that

$$\begin{tikzcd}
k^n \arrow[d, "C_{\beta^*}"'] & k^m \ni e_i \arrow[l, "M_{A^T}"'] \arrow[d, "C_{\gamma^*}"] \\
V^* & W^* \arrow[l, "F^*"]
\end{tikzcd}$$

commutes, it suffices to check on a basis vector e_i, but

$$(F^* \circ C_{\gamma^*})(e_i) = F^*(\omega_i^*) = \omega_i^* \circ F \quad \in V^*$$

and

$$\begin{aligned}(C_{\beta^*} \circ M_{A^T})(e_i) &= C_{\beta^*}(A^T e_i) \\ &= \textstyle\sum_{j=1}^n (A^T e_i)_j \, v_j^* \\ &= \textstyle\sum_{j=1}^n A_{ij} v_j^*\end{aligned}$$

To compare these vectors in V^* it suffices to evaluate on the basis β, where they agree since

$$(\omega_i^* \circ F)(v_a) = \omega_i^*(F v_a) = A_{ia}$$

$$\left(\textstyle\sum_{j=1}^n A_{ij} \, v_j^*\right)(v_a) = A_{ia}.$$