Cartesian product Given sets X, Y you know

$$X \times Y = \{(x,y) \mid x \in X, y \in Y\}.$$

But what is an orcleved pair (x,y)? It isn't some axiom of set theory that these things exist (it is usually an axiom in type theory, but this is a different story). The ordered pair is <u>defined</u> from more primitive elements. Usually we use the <u>Kuratowski pair</u>

$$(x,y) := \{\{x\}, \{x,y\}\}.$$

$$\boxed{\text{QI}} \text{Prove } (x,y) = (9,b) \iff (x = a \land y = b). \text{ Note that sets are equal} \\ \text{iff. they have the same elements.}$$

Well what about $X \times Y \times Z$? Should we invent a "Kuratowski triple"? No. Let us make a general definition : suppose we have a collection of set $\{X_i\}_{i \in I}$ indexed by a set I. We want elements of $\Pi_{i \in I} X_i$ to be I-indexed families of elements $\pi_i \in X_i$, one for each $i \in I$. That is

$\prod_{i \in I} X_i = \left\{ (x_i)_{i \in I} \mid x_i \in X_i \text{ for all } i \in I \right\}$

But what does (a:) it really mean? What set is it?

Let us do an easy case: $I = \{1, 2, 3\}$. Then an element of $\Pi_{i \in I} X_i$ is just an ordered triple (x_1, x_2, x_3) with $x_i \in X_i$. This we may realize as

$$(x_1, y_2, x_3) := \{ (1, x_1), (2, x_2), (3, x_3) \}$$

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Then more generally, $(x_i)_{i\in\mathcal{I}} := \{ (i_j x_i) \mid i \in \mathcal{I} \}$ (2.1) $\boxed{2} \text{ Rove } (\pi_i)_{i \in I} = (y_i)_{i \in I} \iff \pi_i = y_i \text{ for all } i \in I.$ Then TieIXi is the set of all sets like (2.1) above, i.e. sets S where (i) every element is a pair, the fint entry of which lies in I (ii) for each it I there is a unique pair in S beginning with i. (iii) if $(i, x) \in S$ then $x \in X_i$. We call this set the <u>Cartesian product</u>. [Q3] Prove that $\Pi_{i\in\{1,2\}} X_i$ is bijective to $X_i \times X_2$ (defined by Kuratowshi' pair) Are these literally the same set?

Disjoint union How do I form a set S which contains two copies of a given set, say the interval X = [0,1]? well

$X \cup Y = \{x \mid x \in X \text{ or } x \in Y\}$

so $[0,1] \cup [0,1] = [0,1]$. The ordinary union won't work. <u>Idea</u> we can "colour" one copy of X in some arbitrary way, He. X':= {(0,x) | x \in X}. This is bijective to, but distinct from, X. Then S:= X U X' contains two distinct copies of X. We call S the <u>disjoint union</u> of two copies of X and write $S = X \perp X$. Move generally

$$X \amalg Y := (\{0\} \times X) \cup (\{1\} \times Y)$$

More generally, given a collection $\{X_i\}_{i \in I}$ we "colour" X_i by i in order to separate each X_i from X_j , even if as set $X_i = X_j$, to form the <u>disjoint union</u>

