Tutorial 3 : The circle as a group

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This tutorial is a (re) examination of the circle, as a group. This will be important when we arrive at the Hilbert space $L^2(S^1)$ of functions on S^1.

Def^n A topological group is a set X equipped both as a topological space $(X, \mathcal{T})$ and as a group $(X, \cdot, e)$ such that

(i) The function $\cdot : X \times X \longrightarrow X$ is continuous.

(ii) The function $(-)^{-1} : X \longrightarrow X$ is continuous.

An isomorphism of topological groups is a homeomorphism that is also an isomorphism of groups.

$\boxed{Q1}$ Prove that $(\mathbb{R}, +, 0)$ is a topological group

$\boxed{Q2}$ Prove that $(GL(2,\mathbb{R}), \cdot, I_2)$ is a topological group, where $GL(2,\mathbb{R}) \subseteq M_2(\mathbb{R})$ is the set of invertible matrices. Hence show $SO(2,\mathbb{R})$ is also a topological group. (what about the general case?)

We write $GL(2)$, $SO(2)$ for $GL(2,\mathbb{R})$, $SO(2,\mathbb{R})$.

$\boxed{Q3}$ Prove that the bijection $S^1 \longrightarrow SO(2)$ sending $(\cos\theta, \sin\theta)$ to the rotation matrix R_θ is a homeomorphism, where $S^1 \subseteq \mathbb{R}^2$ is the unit circle. Hence S^1 is a topological group with structure maps

$$S^1 \times S^1 \cong SO(2) \times SO(2) \xrightarrow{\cdot} SO(2) \cong S^1$$

$$S^1 \cong SO(2) \xrightarrow{(-)^{-1}} SO(2) \cong S^1$$

$\boxed{Q4}$ Prove S^1 is isomorphic as a topological group to

$$U(1) = \{ e^{i\theta} \mid \theta \in \mathbb{R} \} \subseteq \mathbb{C}$$

under multiplication, with the induced topology.

$\boxed{Q5}$ Let G be a topological group and $H \subseteq G$ a normal subgroup. Let $\sim$ be the equivalence relation $g \sim g'$ iff. $\exists h \in H$ with $g = hg'$, so that $G/H := G/\sim$. Prove that G/H is a topological group when given the quotient topology.

$\boxed{Q6}$ Prove $\mathbb{R}/\mathbb{Z} \cong S^1$ as topological groups.

Note that S^1 is not "canonically" a group, since nobody tells you where to start measuring your angles from, so it is more accurate to say $SO(2)$, $U(1)$, $\mathbb{R}/\mathbb{Z}$ are topological groups, all homeomorphic to S^1 as spaces.

<u>Solutions</u>

$\boxed{Q1}$ Let $f : \mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}$ be $f(x,y) = x+y$ and $g : \mathbb{R} \longrightarrow \mathbb{R}$ be $g(x) = -x$.
There are both linear hence continuous.

$\boxed{Q2}$ We identify $M_2(\mathbb{R})$ with $\mathbb{R}^4$, giving it a topology, then multiplication

$$\begin{tikzcd} M_2(\mathbb{R}) \times M_2(\mathbb{R}) \arrow[r, "\cdot"] & M_2(\mathbb{R}) \\ \mathbb{R}^8 & \mathbb{R}^4 \end{tikzcd}$$

is in each coordinate a polynomial function, hence continuous.
This restricts to a continuous map on $GL(2,\mathbb{R})$, given the subspace topology:

$$\begin{tikzcd} M_2(\mathbb{R}) \times M_2(\mathbb{R}) \arrow[r, "\cdot"] & M_2(\mathbb{R}) \\ GL(2,\mathbb{R}) \times GL(2,\mathbb{R}) \arrow[u, hook] \arrow[r, "\cdot"] & GL(2,\mathbb{R}) \arrow[u, hook] \end{tikzcd}$$

The function $(-)^{-1} : GL(2,\mathbb{R}) \longrightarrow GL(2,\mathbb{R})$ is continuous since the composite

$$\begin{tikzcd} GL(2,\mathbb{R}) \arrow[r] & GL(2,\mathbb{R}) \arrow[r, hook] & M_2(\mathbb{R}) = \mathbb{R}^4 \end{tikzcd}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

is continuous (we may check this coordinate-wise, and the only thing to check is that $\frac{1}{ad-bc}$ is continuous $U \longrightarrow \mathbb{R}$ where $U \subseteq \mathbb{R}^4$ is where this function is defined, namely $U = GL(2,\mathbb{R})$).
Hence $GL(2,\mathbb{R})$ is a topological group.

Q3 Let $Q: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be multiplication with $Q = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Then the map $S^1 \longrightarrow SO(2)$ makes the diagram (Δ is the diagonal, ι the inclusion)

$$\begin{tikzcd}
S^1 \arrow[r, "\Delta"] \arrow[rrrd, bend right] & S^1 \times S^1 \arrow[r, dash, "\iota \times \iota"] & \mathbb{R}^2 \times \mathbb{R}^2 \arrow[r, dash, "id \times Q"] & \mathbb{R}^2 \times \mathbb{R}^2 = \mathbb{R}^4 \\
 & & & SO(2) \arrow[u, "\text{inclusion}"']
\end{tikzcd}$$

$\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$ $\qquad$ $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

commute, hence is continuous.

We already know it is a bijection. To show it is a homeomorphism we need to show the inverse is continuous, but that's easy! The inverse makes the following diagram commute

$$\begin{tikzcd}
SO(2) \arrow[r, hook] \arrow[d] & \overset{\text{col } 1}{\mathbb{R}^2} \times \overset{\text{col } 2}{\mathbb{R}^2} \arrow[d, "\text{projection } \pi_1 \text{ onto first factor}"] \\
S^1 \arrow[r, hook] & \mathbb{R}^2
\end{tikzcd}$$

and is thus continuous.

Q5 Let $m: G/H \times G/H \longrightarrow G/H$ and $i: G/H \longrightarrow G/H$ be the multiplication and inverse, and let $U \subseteq G/H$ be open, $\rho: G \longrightarrow G/H$ the quotient. We have to show $m^{-1}(U)$, $i^{-1}(U)$ are open. Let a point $([g_1], [g_2]) \in m^{-1}(U) \subseteq G/H \times G/H$ be given. Since

$$G \times G \xrightarrow{m^G} G \xrightarrow{\rho} G/H$$

is continuous, we can find $V, W \subseteq G$ with $g_1 \in V$, $g_2 \in W$ such that

$$(g_1, g_2) \in V \times W \subseteq (\rho \circ m^G)^{-1}(U),$$

$$\text{i.e. } \forall a \in V \ \forall b \in W \quad ab \in \rho^{-1}(U)$$

Observe that $m(h, -) : G \longrightarrow G$ is continuous (why?) so

$$\tilde{V} = \bigcup_{h \in H} hV, \quad \tilde{W} = \bigcup_{h \in H} hW$$

are open (saturated!) subsets of G, and if $a \in V$, $b \in W$ and $h, h' \in H$ then $(ha)(h'b) = hh''ab$ for some $h'' \in H$, but this shows that $(ha)(h'b) \sim ab \sim u$ for some $u \in U$ so $(ha)(h'b) \in \rho^{-1}(U)$ and hence

$$(g_1, g_2) \in \tilde{V} \times \tilde{W} \subseteq (\rho \circ m^G)^{-1}(U)$$

But then in $G/H \times G/H$

$$([g_1], [g_2]) \in \rho(\tilde{V}) \times \rho(\tilde{W}) \subseteq m^{-1}(U)$$

and since $\rho(\tilde{V}) \times \rho(\tilde{W})$ is open this completes the proof. The case of i is done similarly.

$\boxed{Q6}$ The map $\mathbb{R} \longrightarrow \mathbb{C}$, $\theta \mapsto e^{2\pi i \theta}$ is continuous, and gives a surjective continuous map $\mathbb{R} \longrightarrow U(1)$ which factors via a continuous bijection $\mathbb{R}/\mathbb{Z} \longrightarrow U(1)$. This is clearly a homomorphism of groups, and it only remains to check it is open, but this is essentially the same as what we did to check $[0,1]/\sim \, \cong S^1$.