The aim of this tutorial is to make you comfortable with the quotient topology. Recall that if $X$ is a topological space and $\sim$ is an equivalence relation then the quotient $X/\sim$ is the set of equivalence classes with the topology

$$\mathcal{J}_{X/\sim} = \{ U \subseteq X/\sim \mid \rho^{-1}(U) \text{ is open} \}$$

where $\rho : X \rightarrow X/\sim$ is the quotient.

Question: Suppose $f : X/\sim \rightarrow Y$ is a function, $Y$ another topological space. Prove $f$ is continuous iff $f \circ \rho$ is continuous.

Lemma: There is a bijection between open subsets of $X/\sim$ and saturated open subsets of $X$, i.e. open subsets $U \subseteq X$ such that if $x \sim y$ and $x \in U$ then also $y \in U$.

Proof: (Proof A) The map $\mathcal{J}_{X/\sim} \rightarrow \mathcal{J}_X$ sending $T \subseteq X/\sim$ to $\rho^{-1}(T)$ is injective since $T = \rho(\rho^{-1}(T))$, and clearly $\rho^{-1}(T)$ is saturated since if $x \sim y$ and $x \in \rho^{-1}(T)$ then $\rho(y) = \rho(x) \in T$ so $y \in \rho^{-1}(T)$. If $U \subseteq X$ is saturated we claim $U = \rho^{-1}(\rho(U))$. The inclusion $U \subseteq \rho^{-1}(\rho(U))$ is automatic, and for the reverse inclusion if $y \in \rho^{-1}(\rho(U))$ then $\rho(y) \in \rho(U)$, i.e. $\rho(y) = \rho(x)$ for some $x \in U$, thus $x \sim y$ and by saturatedness $y \in U$ as claimed. Hence the image of $\mathcal{J}_{X/\sim} \rightarrow \mathcal{J}_X$ is precisely the set of saturated open sets.
(Proof) Observe that by Lemma 4.6.2 the rows in the following commutative diagram are bijective \( \Sigma \) is Sierpiński.)

\[
\begin{array}{ccc}
\text{cts}(X/\sim, \Sigma) & \xrightarrow{=} & \text{cts}(X/\sim) \\
(-) \circ \rho & \downarrow & \downarrow \rho^{-1} \\
\text{cts}(X, \Sigma) & \xrightarrow{=} & \text{cts}(X)
\end{array}
\]  

So it suffices to prove the LHS vertical map is injective, and its image is the set of characteristic functions of saturated open sets. But \( \rho \) is surjective so injectivity is clear, and \( \chi_U : X \to \Sigma \) is in the image iff. \( \chi_U(x) = \chi_U(y) \) whenever \( x \sim y \) (by the universal property of the quotient), or equivalently \( x \in U \iff y \in U \) whenever \( x \sim y \), which is the definition of saturated.  

For the rest of the tutorial we study the following example \( X = [0,1]^2 \) and \( \sim \) the equivalence relation generated by the following pairs:

\[
\begin{align*}
(0,0) & \sim (\lambda,0) \quad 0 \leq \lambda \leq 1 \\
(0,1) & \sim (\lambda,1) \quad 0 \leq \lambda \leq 1 \\
(0,\mu) & \sim (1,\mu) \quad 0 \leq \mu \leq 1 \\
(\lambda,0) & \sim (1,\mu) \quad 0 \leq \mu \leq 1 \\
(\lambda,1) & \sim (0,\mu) \quad 0 \leq \mu \leq 1 \\
(\lambda,0) & \sim (0,\mu) \quad 0 \leq \lambda \leq 1 \\
(0,0) & \sim (1,1) \\
(0,1) & \sim (1,0) \\
(1,0) & \sim (0,1)
\end{align*}
\]

One checks that \( \sim \) is the set

\[
\begin{align*}
\{ & (\lambda,0), (\lambda,1) \mid 0 \leq \lambda \leq 1 \\
\cup & (\lambda,1), (\lambda,0) \mid 0 \leq \lambda \leq 1 \\
\cup & (0,\mu), (1,\mu) \mid 0 \leq \mu \leq 1 \\
\cup & (0,\mu), (1,\mu) \mid 0 \leq \mu \leq 1 \\
\cup & (\lambda,0), (1,\mu) \mid 0 \leq \lambda \leq 1 \\
\cup & (0,0), (0,1), (1,0), (1,1) \\
\}
\]
Let us study the saturated open subsets of $X$ with respect to $\sim$.

Any open set $U \subseteq [0,1]^2$ which does not meet the boundary is saturated. In particular $T = (a,b) \times (c,d)$ is saturated provided $0 < a < b < 1$, $0 < c < d < 1$.

The open set $[0,b) \times (c,d)$ is not saturated since $(0,x) \in U$ for $c < x < d$ but $(1,x) \notin U$.

The open sets

$U = [0,b) \times (c,d) \cup (a,1] \times (c,d)$

$V = (r,s) \times [0,t) \cup (r,s) \times (u,1]$ are saturated $(0 < r < s < 1)$.

What about the corners? An open square at any one, two or three corners cannot be open on its own.
The open set

\[ W = [0,a) \times [0,b) \]
\[ \cup [0,a) \times (c,1) \]
\[ \cup (d,1] \times [0,b) \]
\[ \cup (d,1] \times (c,1] \]

is saturated.

The open sets \( T_a,b,c,d \), \( U_a,b,c,d \), \( V_r,s,t,u \) and \( W_a,b,c,d \) form a basis for the set of saturated open sets, and thus induce a basis for the topology on \( X/\sim \) (actually a smaller subset will do, can you see why?).

The quotient \( X/\sim = [0,1]^2/\sim \) is of course (homeomorphic to) the torus.
Q3 Prove that $X/\sim \cong S^1 \times S^1$ where $S^1 \subseteq \mathbb{R}^2$ is the unit circle.

So we construct a bijective continuous map between these spaces. The idea is that you should always let universal properties do as much of the heavy lifting as you can: so we prefer to construct $X/\sim \to S^1 \times S^1$ rather than $S^1 \times S^1 \to X/\sim$.

Consider $f, g : [0, 1]^2 \to \mathbb{R}^2$ given by

$$f(x, y) = \left( \cos(2\pi x), \sin(2\pi x) \right)$$
$$g(x, y) = \left( \cos(2\pi y), \sin(2\pi y) \right)$$

both are continuous, and induce continuous maps $f, g : [0, 1]^2 \to S^1$

and hence by the universal property of the product, a continuous map

$$\tilde{\Phi} : [0, 1]^2 \to S^1 \times S^1$$

defined by

$$\tilde{\Phi}(x, y) = \left( f(x, y), g(x, y) \right).$$

Since $\tilde{\Phi}$ identifies pain related under $\sim$, we get by the universal property of the quotient a continuous map

$$\Phi : [0, 1]^2/\sim \to S^1 \times S^1$$

It is easy to check it is bijective. Let the inverse function be $\Psi$, so we have only to show $\Psi$ is continuous. But for this it suffices to show that the preimage under $\Psi$ of any element of our chosen basis is open.
For this it is actually convenient to use the arc length metric $d_a$ on $S^1$ with open balls $B^d_a(0) \subseteq S^1$, since for example

$$\Psi^{-1}(T_{x-\epsilon,x+\epsilon,y-\eta,y+\eta}) = \Phi(T_{x-\epsilon,x+\epsilon,y-\eta,y+\eta})$$

$$= \Phi((x-\epsilon,x+\epsilon) \times (y-\eta,y+\eta))$$

$$= B^d_a(p) \times B^d_a(q) \subseteq S^1 \times S^1$$

which is open, as shown below.

The other cases are handled similarly. \(\square\)

**Q4:** (The Torus represents "bi-periodic" functions) Prove that for any space $Y$ there is a bijection between continuous functions $f : \mathbb{R} \times \mathbb{R} \rightarrow Y$ which are bi-periodic, in the sense that there exist $P_1, P_2 > 0$ with

$$f(x, y) = f(x + P_1, y) \quad \forall x, y \in \mathbb{R}$$
$$f(x, y) = f(x, y + P_2) \quad \forall x, y \in \mathbb{R}$$

and continuous functions from the torus $S^1 \times S^1$ to $Y$. 