From Lecture 18 we know how to construct $L^2(X, \mathbb{R})$ for an integral pair $(X, \mu)$ as a set of equivalence classes of Cauchy sequences $(f_n)_{n=0}^\infty$ where each $f_n: X \to \mathbb{R}$ is continuous and the metric used to define Cauchy-ness is the one derived from the $L^2$-norm

$$\| f \|_2 = \left\{ \int_X |f|^2 \right\}^{1/2}$$

We know $L^2(X, \mathbb{R})$ is a Banach space and $(\text{Cts}(X, \mathbb{R}), \| \cdot \|_2) \hookrightarrow (L^2(X, \mathbb{R}), \| \cdot \|_2)$ as normed spaces, but $L^2$ space remains mysterious: what are these vectors in $L^2(X, \mathbb{R}) \setminus \text{Cts}(X, \mathbb{R})$? Are they just some formal objects or do they have some deeper significance? To examine this question we return to the topic of Tutorial 10.

**Example 18-2** Consider $X = [0, 1]$ and the sequence of functions $f_n: X \to \mathbb{R}$ given for $n \geq 4$ by

$$f_n(x) = \begin{cases} 
0 & 0 \leq x < \frac{1}{2} - \frac{1}{n} \\
\frac{1}{n} \left( x - \frac{1}{2} + \frac{1}{n} \right) & \frac{1}{2} - \frac{1}{n} < x < \frac{1}{2} + \frac{1}{n} \\
1 & \frac{1}{2} + \frac{1}{n} \leq x \leq 1
\end{cases}$$

The sequence $(f_n)_{n=4}^\infty$ converges pointwise to $f(x) = \begin{cases} 
1 & x > \frac{1}{2} \\
\frac{1}{2} & x = \frac{1}{2} \\
0 & x < \frac{1}{2}
\end{cases}$

We proved $(f_n)_{n=4}^\infty$ is Cauchy in $(\text{Cts}(X, \mathbb{R}), d_2)$ but does not converge. Hence

$$\hat{f} := [ (f_n)_{n=4}^\infty ] \in L^2(X, \mathbb{R}) \setminus \text{Cts}(X, \mathbb{R})$$

$\exists g \in \text{Cts}(X, \mathbb{R})$

$$\left( f_n \right)_{n=4}^\infty \sim (g)_{n=4}^\infty$$

$d(f_n, g) \to \infty$
We will prove \( L^2(X, \mathbb{R}) \) is a Hilbert space \( \langle \cdot, \cdot \rangle : L^2(X, \mathbb{R}) \times L^2(X, \mathbb{R}) \to \mathbb{R} \) is continuous in each variable and for \( g, h \in C_b(X, \mathbb{R}) \)

\[
\langle g, h \rangle = \int_X gh.
\]

Hence for \( g \in C_b(X, \mathbb{R}) \) \( L^2(X, \mathbb{R}) \setminus C_b(X, \mathbb{R}) \)

\[
\langle g, \hat{f} \rangle = \langle g, \lim_{n \to \infty} f_n \rangle \\
= \lim_{n \to \infty} \langle g, f_n \rangle \\
= \lim_{n \to \infty} \int_X g f_n \\
= \lim_{n \to \infty} \left[ \int_{\frac{y_2}{h_2} - \frac{1}{h_2}}^{\frac{y_2}{h_2} + \frac{1}{h_2}} g \left\{ \frac{1}{h_2} (x - \frac{y_2}{h_2} + \frac{1}{h_2}) \right\} + \int_{\frac{y_2}{h_2} + \frac{1}{h_2}}^{1} g \right] \\
= \int_{\frac{1}{h_2}}^{1} g = \int_X g f
\]

Claim: We should think of \( \hat{f} \) as "being" \( f \). Moreover any Riemann integrable function can be represented in this way by a vector in \( L^2(X, \mathbb{R}) \).
There are several things to check before you should believe this

(i) Does \( \hat{f} \) depend on the choice of approximation \( (f_n)_{n=1}^\infty \) of \( f \) by continuous functions?

(ii) How to find \( (f_n)_{n=1}^\infty \) for a general Riemann integrable \( f \)?

(iii) Is the function

\[
\{ \text{Riemann integrable } f \} \rightarrow L^2(X,\mathbb{R})
\]

injective? Surjective?

The key is:

Since this \( u \) is unique, we denote \( u \) by \( \mathcal{O}_x \) and call this the representing element for \( x \).
**Theorem L20-8 (Riesz representation theorem)** Let \((H, \langle \cdot, \cdot \rangle)\) be a Hilbert space. If \(\mathcal{L} : H \to \mathbb{F}\) is continuous and linear there exists a unique vector \(u \in H\) with

\[ \mathcal{L} = \langle -, u \rangle. \]

**Strategy.** Let \((X, S_X) = ([0, 1], \int_0^1)\).

1. Let \(f : X \to \mathbb{R}\) be Riemann integrable.
2. Question: \(\hat{f} \in L^2(X, \mathbb{R})\) “representing” \(f\).
3. \(H = (L^2(X, \mathbb{R}), \langle -, \cdot \rangle)\), \(\mathcal{L} : H \to \mathbb{F}\) given \(\mathcal{L} \in L^2(X, \mathbb{R})\).

**Lemma** Given \(f\) integrable consider the function

\[ \mathcal{L} : Cts(X, \mathbb{R}) \to \mathbb{R}, \quad \mathcal{L}(g) = \int_0^1 g f \, dx. \]

is continuous and linear with respect to \(\mathcal{L}\), hence \(\mathcal{L} \in Cts(X, \mathbb{R})\).

**Proof.** Linearity is clear. For continuity observe that

\[
\left| \int g f \, dx - \int g' f \, dx \right| = \left| \int (g-g') f \, dx \right| \\
\mathcal{L}(g) - \mathcal{L}(g') \leq \int |g-g'| |f| \, dx \quad \| (g-g') f \|_1 \\
\text{Holden inequality} \quad \leq \| g - g' \|_2 \| f \|_2 \\
\text{Hence } \mathcal{L} \text{ is bounded } \| \mathcal{L} \| \leq \| f \|_2. \quad \Box
\]
Let \( c : C^0(\mathbb{X}, \mathbb{R}) \to L^2(\mathbb{X}, \mathbb{R}) \) be the inclusion. Since \( \gamma : C^0(\mathbb{X}, \mathbb{R}) \to \mathbb{R} \) is continuous and linear there is a unique continuous linear \( \gamma' \) making

\[
\begin{array}{ccc}
L^2(\mathbb{X}, \mathbb{R}) & \xrightarrow{\gamma'} & \mathbb{R} \\
\uparrow & \searrow \gamma & \\
C^0(\mathbb{X}, \mathbb{R})
\end{array}
\]

commute. Note \( \gamma' \) is continuous.

By the Riesz representation theorem there is a unique representing element \( \hat{f} \) for \( \gamma' \). That is, \( \hat{f} \in L^2(\mathbb{X}, \mathbb{R}) \) and \( \langle \gamma' \hat{f} \rangle = \gamma' \).

Equivalently

\[
\hat{f} = \left\{ (g_n)_{n=0}^{\infty} \right\} \quad \langle g, \hat{f} \rangle = \int_{0}^{1} g \hat{f} \, dx \quad \forall g \in C^0(\mathbb{X}, \mathbb{R})
\]

Lemma

\[
\| \hat{f} \|_2 = \| f \|_2 = \left\{ \int_{0}^{1} | f(x) |^2 \, dx \right\}^{\frac{1}{2}}
\]

\[
\lim_{n \to \infty} \| f_n \|_2 = \left\{ \text{Riemann integrable} \right\} \xrightarrow{\text{a.e.}} L^2(\mathbb{X}, \mathbb{R})
\]

- \( \Phi \) is not injective, \( \ker(\Phi) = \{ \text{almost everywhere zero functions} \} \).
- \( \Phi \) is not surjective either!

\[
\left\{ \text{Lebesgue integrable} \right\} \xrightarrow{\text{a.e. equality}} L^2(\mathbb{X}, \mathbb{R})
\]