The axiom of choice, in the form of Zorn's lemma, plays a foundational role in much of mathematics, and often in gruesomely technical ways. One important application is to show that for a field $k$, and vector space $V$ over $k$, any linearly independent set $C \subseteq V$ (possibly empty) is contained in a basis $\mathcal{B}$ of $V$ (i.e., $C \subseteq \mathcal{B}$). We say: any linearly independent set may be extended to a basis. In particular, taking $C = \emptyset$, this shows that any vector space has a basis (which is not clear a priori unless $V$ can be spanned by a finite set, in which case it is easy).

We begin with the statement of the axiom of choice. This does not require (or have!) a proof: that is why it is called an axiom. We assume it.

**Axiom of Choice (AC)** If $\{X_i\}_{i \in I}$ is an indexed family of sets and $X_i \neq \emptyset$ for each $i \in I$, then $\prod_{i \in I} X_i \neq \emptyset$.

Recall an indexed family $\{X_i\}_{i \in I}$ is just a surjective function $f : I \to A$ with domain $I$, and $f(i) = X_i \subseteq A$, and

$$\prod_{i \in I} X_i = \{f : I \to UA \mid f(i) \subseteq X_i \text{ for all } i \in I\}.$$

Here $UA$ is the union set with $x \in UA \iff \exists a \in A (x \in a)$, or if you like, visualise $UA$ as obtained by "dissolving" the boundaries between every element of $UA$. A function $f \in \prod_{i \in I} X_i$ chooses one element per $i$:

\begin{align*}
x_1, x_2 \in A \\
x', x'' \in X_1 \\
x, x' \in X_2\end{align*}
Def. Let \((\mathcal{P}, \leq)\) be a partially ordered set. Then a subset \(Q \subseteq \mathcal{P}\) is called a chain if \(\forall x, y \in Q (x \leq y \lor y \leq x)\). The empty set is vacuously a chain.

Def. Let \((\mathcal{P}, \leq)\) be a partially ordered set, \(Q \subseteq \mathcal{P}\). An element \(x \in \mathcal{P}\) is an upper bound for \(Q\) if \(\forall y \in Q (y \leq x)\). An element \(m \in \mathcal{P}\) is maximal if \(\forall y \in \mathcal{P} (m \leq y \Rightarrow m = y)\).

Example. In the power set \(\mathcal{P}(\{0, 1\})\) with inclusion, the subset \(Q = \{\{0\}, \{1\}\}\) is not a chain.

Zorn's Lemma (ZL). If every chain in \((\mathcal{P}, \leq)\) has an upper bound, then \(\mathcal{P}\) contains a maximal element.

Proof. By transfinite induction, which requires the language of ordinals and this does not fit in a tutorial. You can see the "Basic Set Theory" notes on my webpage for a self-contained presentation from scratch. Since Zorn's Lemma is equivalent (given the other axioms) to AC you are safe in taking the attitude that Zorn's Lemma is an axiom in your personal universe. □

The statement of AC (which "obviously should be true") is included here in order to bully you into believing ZL (which looks less "obvious") via \(AC \iff ZL\).

Theorem. Let \(k\) be a field and \(V\) a vector space over \(k\). If \(\mathcal{C} \subseteq V\) is linearly independent then there exists a basis \(\mathcal{B}\) for \(V\) with \(\mathcal{B} \supseteq \mathcal{C}\).

Proof. Let \(\mathcal{P}\) be the following set, partially ordered by inclusion
\[ P = \{ \mathbf{x} \in V \mid \mathbf{x} \text{ is linearly independent and } \mathbf{x} \in C \} \]

Note that \( C \in P \) so this set is nonempty (\( C = \emptyset \) is allowed). We claim every chain in \( P \) has an upper bound. Let \( Q \subseteq P \) be a chain. Then \( UQ \subseteq V \) and we claim this is a linearly independent set. Given \( \{ \mathbf{v}, \ldots, \mathbf{v}_n \} \subseteq UQ \), we have \( \mathbf{v}_i \in \mathbf{x}_i \) for some \( \mathbf{x}_i \in Q \), and by induction and the assumption that \( Q \) is a chain we have that one of the \( \mathbf{x}_i \), say \( \mathbf{x}_{i_0} \), contains all the others. But then \( \{ \mathbf{v}, \ldots, \mathbf{v}_n \} \subseteq \mathbf{x}_{i_0} \) must be LI. A set is LI iff. all its finite subsets are LI, so \( UQ \) is LI. Since \( \mathbf{x} \in UQ \), whenever \( \mathbf{x} \in Q \) this shows \( Q \) has an upper bound.

Hence by Zorn's Lemma \( P \) contains a maximal element \( Q^{\max} \). Suppose \( \text{span}(Q^{\max}) \neq V \), say \( w \in V \setminus \text{span}(Q^{\max}) \). Then \( \{w\} \cup Q^{\max} \) is LI and hence \( \{w\} \cup Q^{\max} \in P \), but since
\[ \{w\} \cup Q^{\max} \supseteq Q^{\max} \]
this contradicts maximality of \( Q^{\max} \). This contradiction shows \( Q^{\max} \) is a basis for \( V \), and by construction it contains \( C \). \( \square \)

**Corollary** Every vector space has a basis.

**Proof** \( C = \emptyset \). \( \square \)
Addendum: (Secret uses of AC)

Let \((X, d)\) be a metric space, \(U \subseteq X\) an open subset, and for every \(x \in U\) let \(\varepsilon_x > 0\) be such that \(B_{\varepsilon_x}(x) \subseteq U\).

You just used the axiom of choice! The assignment \(x \mapsto \varepsilon_x\) is an element of

\[
\bigsqcup_{x \in U} \{ \varepsilon \in \mathbb{R}^+_0 \mid B_{\varepsilon}(x) \subseteq U \}.
\]

Each of the sets \(\{ \varepsilon \mid B_{\varepsilon}(x) \subseteq U \}\) is nonempty and you used that the product is nonempty, that is, you could simultaneously choose one \(\varepsilon\) for each \(x\). This is the axiom of choice.

Suppose you could prove that for each \(x\) there was a unique \(\varepsilon_x > 0\) satisfying \(B_{\varepsilon_x}(x) \subseteq U\) and some other property \(\phi(x, \varepsilon)\) (this stands for a formula in which \(x, \varepsilon\) occur). The formula

\[
\Xi(x, \varepsilon) : \quad \varepsilon \in \mathbb{R}^+_0 \land B_{\varepsilon}(x) \subseteq U \land \phi(x, \varepsilon)
\]

determines, by the Axiom of Comprehension, a subset

\[
\{ (x, \varepsilon) \in U \times \mathbb{R}^+_0 \mid \Xi(x, \varepsilon) \text{ holds} \}
\]

and this subset is precisely the graph of the function \(U \to \mathbb{R}^+_0\) assigning to each \(x \in U\) this "special" \(\varepsilon_x\). In this way we avoided invoking AC, but in practice this is not always possible (or convenient).