<u>Solutions</u>

$\boxed{Q6}$ Note that v_i^* is the linear map

$$V \xrightarrow{c_\beta^{-1}} k^n \xrightarrow{e_i^*} k$$

so that there is a well-defined linear map $c_{\beta^*} : k^n \longrightarrow V^*$ sending e_i to v_i^*, and we just need to show this is an isomorphism. But this follows from commutativity of

$$\begin{tikzcd}
k^n \arrow[rr, "c_{\beta^*}"] \arrow[dr, "\cong", "(**)"'] & & V^* \arrow[dl, "\cong"', "(c_\beta)^*"] \\
& (k^n)^* &
\end{tikzcd}$$

which we can verify on the basis $e_1, \ldots, e_n$ by obsewing that

$$\begin{aligned}
(c_\beta)^* c_{\beta^*}(e_i)(e_j) &= c_\beta^*(v_i^*)(e_j) \\
&= [v_i^* \circ c_\beta](e_j) \\
&= [e_i^* \circ c_\beta^{-1} \circ c_\beta](e_j) \\
&= e_i^*(e_j) = \delta_{ij}
\end{aligned}$$

Hence $(c_\beta)^* c_{\beta^*}(e_i) = e_i^*$, as claimed.

$\boxed{Q7}$ To check that

$$\begin{tikzcd} k^n \arrow[d, "C_{\beta^*}"'] & k^m \ni e_i \arrow[l, "M_{A^T}"'] \arrow[d, "C_{\gamma^*}"] \\ V^* & W^* \arrow[l, "F^*"] \end{tikzcd}$$

commutes, it suffices to check on a basis vector e_i, but

$$(F^* \circ C_{\gamma^*})(e_i) = F^*(\omega_i^*) = \omega_i^* \circ F \quad \in V^*$$

and

$$\begin{aligned}(C_{\beta^*} \circ M_{A^T})(e_i) &= C_{\beta^*}(A^T e_i) \\ &= \sum_{j=1}^{n} (A^T e_i)_j \, v_j^* \\ &= \sum_{j=1}^{n} A_{ij} v_j^*\end{aligned}$$

To compare these vectors in V^* it suffices to evaluate on the basis $\mathcal{B}$, where they agree since

$$(\omega_i^* \circ F)(v_a) = \omega_i^*(F v_a) = A_{ia}$$

$$\left(\sum_{j=1}^{n} A_{ij} v_j^*\right)(v_a) = A_{ia}.$$