Note that $v_i^*$ is the linear map

\[ \begin{array}{ccc}
V & \xrightarrow{c_{\rho}^*} & k^n & \xrightarrow{e_i^*} & k \\
\end{array} \]

so that there is a well-defined linear map $c_{\rho}^* : k^n \to V^*$ sending $e_i$ to $v_i^*$, and we just need to show this is an isomorphism. But this follows from commutativity of

\[ \begin{array}{ccc}
k^n & \xrightarrow{c_{\rho}^*} & V^* \\
(k^*) & \cong & (k^n)^* \\
(c_{\rho}^*)^* & \cong & (c_{\rho}^*) \\
\end{array} \]

which we can verify on the basis $e_1, \ldots, e_n$ by observing that

\[
(c_{\rho})^*(c_{\rho}^*)^* (e_i)(e_j) = c_{\rho}^* (v_i^*)(e_j) \\
= [v_i^* \circ c_{\rho}](e_j) \\
= [e_i^* \circ c_{\rho}^{-1} \circ c_{\rho}](e_j) \\
= e_i^*(e_j) = \delta_{ij}
\]

Hence $c_{\rho}^* (c_{\rho}^*)^* = e_i^*$, as claimed.
To check that

\[
\begin{array}{ccc}
\mathbb{R}^n & \xrightarrow{M_{A^T}} & \mathbb{R}^m \\
C_{\beta^*} & \downarrow & C_{\delta^*} \\
V^* & \xleftarrow{F^*} & W^*
\end{array}
\]

commutes, it suffices to check on a basis vector \(e_i\), but

\[
(F^* \circ C_{\delta^*})(e_i) = F^*(\omega_i^*) = \omega_i^* \circ F \in V^*
\]

and

\[
(C_{\beta^*} \circ M_{A^T})(e_i) = C_{\beta^*}(A^T e_i) = \sum_{j=1}^n (A^T e_i)_j v_j^* = \sum_{j=1}^n A_{ij} v_j^*
\]

To compare these vectors in \(V^*\) it suffices to evaluate on the basis \(\beta\), where they agree since

\[
(\omega_i^* \circ F)(v_a) = \omega_i^*(F v_a) = A_{ia}
\]

\[
(\sum_{j=1}^n A_{ij} v_j^*)(v_a) = A_{ia}.
\]