Example 11.8-2 Consider $X = [0, 1]$ and the sequence of functions $f_n : X \rightarrow \mathbb{R}$ given for $n \geq 4$ by

$$f_n(x) = \begin{cases} 
0 & 0 \leq x \leq \frac{1}{2} - \frac{1}{n} \\
\frac{1}{n} (x - \frac{1}{2} + \frac{1}{n}) & \frac{1}{2} - \frac{1}{n} < x < \frac{1}{2} + \frac{1}{n} \\
1 & \frac{1}{2} + \frac{1}{n} \leq x \leq 1 
\end{cases}$$

The sequence $(f_n)_{n=4}^{\infty}$ converges pointwise to

$$f(x) = \begin{cases} 
1 & x > \frac{1}{2} \\
\frac{1}{2} & x = \frac{1}{2} \\
0 & x < \frac{1}{2} 
\end{cases}$$

but this convergence is certainly not uniform (as the uniform limit of continuous functions is continuous). So $(f_n)_{n=4}^{\infty}$ does not converge in $(\text{Cts}(X, \mathbb{R}), d_{\infty})$ (if it converged, it would have to be to $f$) and hence is not Cauchy (as this space is complete). However we claim the sequence is Cauchy in $(\text{Ch}(X, \mathbb{R}), d_p)$ but still does not converge, where throughout $1 \leq p < \infty.$
Observe that for $m \geq n \geq 4$ and $1 \leq p < \infty$, $\|f_m - f_n\|_p \leq \|f - f_n\|_p$ so

$$d_p(f_m, f_n) = \|f_m - f_n\|_p = \left\{ \int_x |f_m - f_n|^p \right\}^{1/p}$$

$$\leq \left\{ \int_x |f - f_n|^p \right\}^{1/p}$$

$$= \left\{ 2 \int_{\frac{1}{2}}^{\frac{1}{2} + \frac{n}{2}} |f - f_n|^p \right\}^{1/p}$$

$$= \left\{ 2 \int_{\frac{1}{2}}^{\frac{1}{2} + \frac{n}{2}} \left( 1 - \frac{n}{2} (x - \frac{1}{2} + \frac{n}{2}) \right)^p \right\}^{1/p}$$

$$u = 1 - \frac{n}{2} (x - \frac{1}{2} + \frac{n}{2})$$

$$= 2^{1/p} \left\{ \frac{2}{n} \int_0^{1/2} u^p du \right\}^{1/p}$$

$$= 2^{1/p} n^{-1/p} \left\{ \int_0^{1/2} u^p du \right\}^{1/p}$$

$$> 0$$
Hence \((f_n)_{n=1}^\infty\) is Cauchy in \((C^p(X,\mathbb{R}),d_p)\). Now we claim this sequence does not converge in \((C^p(X,\mathbb{R}),d_p)\). To say \(f_n \rightarrow g\) w.r.t. \(d_p\) says that \(\forall \varepsilon > 0 \exists N \forall n \geq N \quad \int_X |f_n - g|^p < \varepsilon^p\).

**Problem** \(\int_X |f_n - g|^p \) small \(\Rightarrow\) \(f_n(x) - g(x)\) small any given \(x\).

Suppose for a contradiction that \(f_n \rightarrow g\) in \((C^p(X,\mathbb{R}),d_p)\). Then since the restriction function for \([c,d] \subseteq (\frac{1}{2},1]\) i.e. \((C^p(X,\mathbb{R}),d_p) \rightarrow (C^p([c,d],\mathbb{R}),d_p)\) is continuous so \(f_n \big|_{[c,d]} \rightarrow g \big|_{[c,d]}\), i.e.

\[\forall \varepsilon > 0 \exists N \forall n \geq N \left( \left\{ \int_c^d |f_n - g|^p \right\}^{\frac{1}{p}} < \varepsilon \right)\]

Claim that \(\|1 - g\|_p < \varepsilon\) for any positive \(\varepsilon\) and hence \(\|1 - g\|_p = 0\). To see this let \(\varepsilon\) be given and find \(N\) such that \(\|f_n - g\|_p < \varepsilon\) for \(n \geq N\) and \(f_n \big|_{[c,d]} = 1\) for \(n \geq N\). Thus \(\|1 - g\|_p [c,d] < \varepsilon\), as required.

But since \(\|\cdot\|_p\) is a norm \(1 - g = 0\) as claimed.

(i) since \([c,d]\) was arbitrary \(g(x) = 1\) for \(x > \frac{1}{2}\).

(ii) similarly \(g(x) = 0\) for \(x < \frac{1}{2}\).

This shows \(g\) is not continuous, a contradiction.

**Exercise L18-7** Give a counterexample to show that convergence \(f_n \rightarrow f\) in \((C^p([a,b],\mathbb{R}),d_p)\) for \(1 \leq p < \infty\) does not imply pointwise convergence.

(Hint: \(\square\)) (However \(f_n \rightarrow f\) does imply pointwise convergence "almost everywhere" in a precise sense we will define later).