Lecture 11: Hausdorff spaces

So far we have two main sources of examples of topological spaces: the underlying set of a metric space with the metric topology, and finite CW-complexes (these two classes, of course, have significant overlap).

We have seen a few other examples like the Sierpiński space

$$
\Sigma = \{0, 1\} \quad T_\Sigma = \{\emptyset, \Sigma, \{1\}\}
$$

but these examples seem rather “exotic.” One way in which $\Sigma$ is strange is that the point $1 \in \Sigma$ is not closed (since its complement $\{0\}$ is not open).

Clearly any point in a metric topology is a closed set, and “non-closed point” seems like an oxymoron. In many parts of mathematics (with important exceptions) one is only interested in spaces in which all points are closed, and even better, any two distinct points may be separated by open neighborhoods:

**Def** A topological space $X$ is **Hausdorff** if for any pair $x, y \in X$ of distinct points there exist open $U, V \subseteq X$ with $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

**Lemma LII-1** If $X$ is Hausdorff and $x \in X$ then $\{x\}$ is closed.

**Proof** For $y \in X \setminus \{x\}$ let $V_y \subseteq X$ be open with $x \notin V_y$, $y \in V_y$. Then $X \setminus \{x\} = \bigcup V_y$. ☐
Lemma LII-2 If \( X \) is metrisable then it is Hausdorff.

**Proof** Say the topology is induced by a metric \( d \). Given \( x, y \in X \) distinct \( d(x, y) > 0 \) and for any \( \varepsilon \leq \frac{1}{2} d(x, y) \) the balls \( U = B_\varepsilon(x) \), \( V = B_\varepsilon(y) \) will do, since

\[
2\varepsilon \leq d(x, y) \leq d(x, z) + d(z, y)
\]

and so if \( z \in U \), so \( d(z, x) < \varepsilon \) then \( d(z, y) > 2\varepsilon - \varepsilon = \varepsilon \), so \( z \notin V \). \( \square \)

Needless to say, \( \Sigma \) is not Hausdorff, and hence not metrisable.

**Exercise LII-1** Give an example of a space \( X \) in which points are closed, but which is not Hausdorff.

**Exercise LII-2** Prove any subspace of a Hausdorff space is Hausdorff.

There are some obvious questions: which of the standard constructions on topological spaces (product, disjoint union, quotient) preserve the Hausdorff condition? And then, who cares? What is this condition "good for"?

Lemma LII-3 If \( X, Y \) are Hausdorff then so is \( X \times Y \).

**Proof** If \( (x_1, y_1), (x_2, y_2) \in X \times Y \) are distinct, without loss of generality \( x_1 \neq x_2 \). Then there exist open disjoint \( U, V \subseteq X \) with \( x_1 \in U \), \( x_2 \in V \), and then \( U \times Y \), \( V \times Y \subseteq X \times Y \) are open, disjoint and \( (x_1, y_1) \in U \times Y \), \( (x_2, y_2) \in V \times Y \). \( \square \)

**Exercise LII-4** Prove that if \( \{ X_i \}_{i \in I} \) is a family of Hausdorff spaces that \( \bigcap_{i \in I} X_i \) is Hausdorff.
Exercise LIII-5  Prove if $X, Y$ are Hausdorff that $X \sqcup Y$ is Hausdorff.

Example LIII-1  Consider the pushout

\[
\begin{array}{ccc}
\mathbb{R} \setminus \{0\} & \xrightarrow{l} & \mathbb{R} \\
\downarrow & & \downarrow f \\
\mathbb{R} & \xrightarrow{g} & X = (\mathbb{R} \sqcup \mathbb{R})/\sim
\end{array}
\]

where $l$ is the inclusion. Here $X$ is obtained from two copies of the line glued along every point but the origin. The space $X$ is often called the "line with doubled origin". Let us write $O_1 := f(0)$ and $O_2 := g(0)$. Then as a set

\[X = \mathbb{R} \setminus \{0\} \cup \{O_1, O_2\}\]

The open neighborhoods of $O_1$ in $X$ are the images under the quotient map of open sets $U \sqcup V \subseteq \mathbb{R} \sqcup \mathbb{R}$ which are "saturated", i.e. closed under $\sim$, and for which $O \notin U$. But such an open set must contain points in the second copy of $\mathbb{R}$ arbitrarily close to $O$, and thus in $X$ there can be no open neighborhood of $O_1$ avoiding every neighborhood of $O_2$.

Hence $X$ is not Hausdorff, and so the quotient of a Hausdorff space need not be Hausdorff (as $\mathbb{R} \sqcup \mathbb{R}$ is Hausdorff).
Exercise LII-6 If \( X \cong Y \) and \( X \) is Hausdorff then so is \( Y \).

The Hausdorff condition is very useful, as we will see. But here are two immediately useful consequences:

**Lemma LII-5** Any compact subspace of a Hausdorff space is closed.

**Proof** Say \( K \subseteq X \) is compact, \( X \) Hausdorff. Given \( x \notin K \) choose for each \( k \in K \) a pair of disjoint open sets \( U_k, V_k \) with \( k \in U_k \) and \( x \in V_k \). The \( \{U_k\}_{k \in K} \) cover \( K \), and since it is compact finitely many, say \( \{U_{k_1}, \ldots, U_{k_n}\} \) will do. But then \( x \in V_{k_1} \cap \cdots \cap V_{k_n} \) is an open neighborhood of \( x \) disjoint from \( K \).

This proves \( K \) is closed. \( \Box \)

**Lemma LII-6** Suppose \( X \) is compact and \( Y \) is Hausdorff. Then any continuous bijection \( f : X \rightarrow Y \) is a homeomorphism.

**Proof** It suffices to show that if \( U \subseteq X \) is open then \( f(U) \subseteq Y \) is open. But \( X \setminus U \) is closed and therefore compact (Ex. L9-5) and so (by Prop L9-3)

\[
f(X \setminus U) = Y \setminus f(U)
\]

is compact, and therefore closed by Lemma LII-5. Hence the complement \( f(U) \) must be open. \( \Box \)

Exercise LII-7 (i) If \( \sim \) is an equivalence relation on a space \( X \), and \( X/\sim \) is the quotient, prove there is a bijection between open set of \( X/\sim \) and open subset \( U \) of \( X \) which are saturated, i.e. \( x \sim y \) and \( x \in U \) implies \( y \in U \).
(ii) Let $E$ be a nonempty set, $R \subseteq E \times E$ a subset, and define $\tilde{R}$ to be the set of all pairs $(e,e')$ such that there exists a sequence $e = e_0, e_1, \ldots, e_n = e'$ \[ n \geq 0 \]

with $e_i \in E$ and for each $0 \leq i < n$ either $(e_i, e_{i+1}) \in R$ or $(e_{i+1}, e_i) \in R$. Prove $\tilde{R}$ is the smallest equivalence relation containing $R$. (the $n=0$ case says $(e,e) \in \tilde{R}$ for all $e \in E$).

**Def** A continuous map $f: X \to Y$ is open if whenever $U \subseteq X$ is open so is $f(U)$.

**Theorem 11.4.** Any finite CW-complex $X$ is compact Hausdorff.

**Proof.** The compactness was Theorem 11.5. We prove $X$ is Hausdorff by induction on a presentation of $X$ as $X_0, X_1, \ldots, X_n = X$. Since $X_0$ is finite and discrete it is Hausdorff (every point is its own open neighborhood). Suppose we have shown $X_{i-1}$ is Hausdorff. The space $X_i$ is constructed from $X_{i-1}$ by a pushout:

$$
\begin{array}{ccc}
\coprod_{\alpha \in \Lambda} S^{\ell-1} & \xrightarrow{f} & X_{i-1} \\
\downarrow & & \downarrow \\
\coprod_{\alpha \in \Lambda} D^\ell & \xrightarrow{\text{label the copies of } S^{\ell-1}, D^\ell} & X_i = (X_{i-1} \amalg \coprod_{\alpha} D^\ell)/\sim
\end{array}
$$

Now a generic quotient will not be Hausdorff, so we need a special argument unique to this quotient. We use the following observations, which all follow from Exercise 11.7. Let $\rho: X_{i-1} \amalg \coprod_{\alpha} D^\ell \to X_i$ be the quotient.

We drop the $\coprod$ and view $\coprod_{\alpha} S^{\ell-1}$ as a subspace of $\coprod_{\alpha} D^\ell$ (as it is!)
(0) If $x, y \in X_{i-1} \cup \bigcup \alpha D_i^i$ are distinct points and $\rho(x) = \rho(y)$ then there is a sequence $x = e_0, e_1, \ldots, e_n = y$ with $0 \leq j < n$ either

\[(e_j, e_{j+1}) \text{ or } (e_{j+1}, e_j) \in \left\{ \left( (f(s), s) \right) \mid s \in \bigcup \alpha s^{i-1} \right\} \]

In particular if $x, y \in X_{i-1} \setminus f(\bigcup \alpha s^{i-1})$ then $\rho(x) \neq \rho(y)$, and if $x, y \in \bigcup \alpha D_i^i \setminus \bigcup \alpha s^{i-1}$ then $\rho(x) \neq \rho(y)$. In fact

\[
\sim = \left\{ (u, v) \mid u \in X_{i-1} \cup \bigcup \alpha D_i^i \right\} \cup \left\{ (s, f(s)), (f(s), s) \mid s \in \bigcup \alpha s^{i-1} \right\}
\cup \left\{ (x, y) \mid x, y \in \bigcup \alpha s^{i-1} \text{ such that } f(x) = f(y) \right\}
\]

(1) $f(\bigcup \alpha s^{i-1}) = \bigcup \alpha f(s^{i-1})$ is compact (PropL9-3) hence closed (using Lemma L11-5 and the inductive hypothesis that $X_{i-1}$ is Hausdorff).

(2) The map $X_{i-1} \rightarrow X_i$ is injective and the restriction to the complement of $\bigcup \alpha f(s^{i-1})$ is open.

(3) The restriction of $D^i_{(\alpha)} \rightarrow \bigcup \alpha D^i_{(\alpha)} \rightarrow X_i$ to the open disk $B^i \subseteq D^i$ is injective and open, where $B^i = \{ x \in \mathbb{R}^i \mid \| x \| < 1 \}$.
Let \( x, y \in X_i \) be distinct.

**Easy cases** \( x, y \) both lie in the image of one of the injective open maps from (2), (3) above (perhaps different maps).

Since \( B^i, X_i - I \) are Hausdorff and these maps are open, it is clear we can find open neighborhoods of \( x, y \) which are disjoint.

Otherwise at least one of the points lies in the image of \( \amalg \alpha S^{i-1}_\alpha \), say \( x \).

Now \( \rho^{-1}(\alpha) \) contains precisely one point \( x_0 \) of \( X_i - I \), and for each \( \alpha \) some closed subset \( C(\alpha) := \rho^{-1}(\alpha) \cap D_i^c(\alpha) \) of the boundary sphere (maybe \( C(\alpha) = \emptyset \), but it can be nonempty for multiple \( \alpha \) and is nonempty for at least one \( \alpha \)). Given an open neighborhood \( U \) of \( x_0 \) in \( X_i - I \) each \( f_x^{-1}(U) \subseteq S^{i-1}_\alpha \) is an open neighborhood of \( f_x^{-1}(x_0) \).
Since $S_{(x)}^{\epsilon^{-1}} \subseteq D^{\epsilon}(\alpha)$ has the subspace topology, we can find an open
subset $U\alpha \subseteq D^{\epsilon}(\alpha)$ such that $U\alpha \cap S_{(x)}^{\epsilon^{-1}} = f^{-1}_{\alpha}(V)$. Then

$$\tilde{U} := U \cup \bigcup_{\alpha \in \Lambda} U\alpha$$

is an open subset of $X_{i-1} \sqcup \bigcup \alpha D^{\epsilon}(\alpha)$ and $\rho^{\epsilon'})\tilde{U} = \tilde{U}$ so $\rho^{\tilde{U}}$ is open,
and it is a neighborhood of $x$ in $X_{i}$. Note we chose $U$ arbitrarily, and $U\alpha$ arbitrarily subject to $U\alpha \cap S_{(x)}^{\epsilon^{-1}} = f^{-1}_{\alpha}(V)$.

**Case I** $y$ also lies in the image of $\bigcup \alpha S_{(x)}^{\epsilon^{-1}}$. Let $y_{0}$ be the unique
preimage in $X_{i-1}$ and find (using $X_{i-1}$ Hausdorff) open
neighborhoods $x_{0} \in U$ and $y_{0} \in V$ s.t. $UNV = \emptyset$. Then
running the above construction also for $y$ and $V$, we have
$f^{-1}_{\alpha}(V) \cap f^{-1}_{\alpha}(V) = f^{-1}_{\alpha}(V \cup V) = \emptyset$ and we may choose
$U\alpha, V\alpha \subseteq D^{\epsilon}(\alpha)$ disjoint with $U\alpha$ as above and
$V\alpha \cap S_{(x)}^{\epsilon^{-1}} = f^{-1}_{\alpha}(V)$ (why? see Ex 11-10)

Then

$$\tilde{V} := V \cup \bigcup_{\alpha \in \Lambda} V\alpha$$

gives an open neighborhood $\rho\tilde{V}$ of $y$ disjoint from $\rho\tilde{U}$.
Case II  y is not in the image of $\mathbb{I} \times S_{(\alpha)}^{-1}$. Since $U \times f(\mathit{S}_{(\alpha)}^{-1})$ is closed we can find an open neighborhood $V$ of $y$ disjoint from $U$ which does not meet $U \times f(\mathit{S}_{(\alpha)}^{-1})$. Then $\rho V$ is an open neighborhood of $y$ disjoint from $\rho V$, completing the proof.

There are some stronger separation axioms that are also in common use:

**Def** Suppose one-point sets are closed in $X$. Then $X$ is called

- **regular** if for each pair of a point $x$ and $B \subseteq X$ closed disjoint from $x$, there exist disjoint open sets containing $x$ and $B$ respectively.

- **normal** if for each pair of disjoint closed sets $A, B \subseteq X$ there exist disjoint open sets containing $A$ and $B$ respectively.

**Remark** It is clear that normal $\Rightarrow$ regular $\Rightarrow$ Hausdorff.

**Exercise L11-8** Prove any metrisable space is normal.

**Exercise L11-9** Prove any compact Hausdorff space is normal.

(Hint: use the proof of Lemma L11-5).

Hence a finite CW-complex is normal.
Exercise LIl-10 (i) For \( i \geq 1, 0 < h < 1 \), prove \( \gamma : S^{i-1} \times [0, h) \rightarrow D^i \) defined by

\[ \gamma(x, \lambda) = (1-\lambda)x \]

gives a homeomorphism of \( S^{i-1} \times (0, h) \) with an open neighborhood of \( S^{i-1} \) in \( D^i \) (usually called a collar).

(ii) Fill in the final detail in the proof of Theorem LIl-4, by showing that if \( C_1, C_2 \subseteq S^{i-1} \) are disjoint closed sets, and \( U_1, U_2 \subseteq S^{i-1} \) are disjoint open sets with \( U_j \supseteq C_j \) there exist disjoint open subsets \( W_1, W_2 \) of \( D^i \) such that \( W_j \cap S^{i-1} = U_j \) for \( j \in \{1, 2\} \).

Exercise LIl-11 (i) Prove \( X \) is Hausdorff if and only if the diagonal

\[ \Delta = \{ (x, x) \in X \times X \mid x \in X \} \]

is a closed subset of \( X \times X \).

(ii) Let \( G \) be a topological group (see Tutorial 4). Prove that \( G \) is Hausdorff if and only if \( \{e\} \) is closed, where \( e \) is the identity element.

(iii) Let \( G \) be a topological group and \( H \subseteq G \) a normal subgroup. Prove \( G/ H \) is Hausdorff if and only if \( H \subseteq G \) is closed.
Exercise LII-12 In the context of Example LII-1, prove that there does not exist a continuous function \( f: X \rightarrow \mathbb{R} \) with \( f(O_1) \neq f(O_2) \).

Aside on the Hausdorff condition

The following statement from Munkres is typical:

"Topologies in which one-point sets are not closed, or in which sequences can converge to more than one point, are considered by many mathematicians to be somewhat strange. They are not really very interesting, for they seldom occur in other parts of mathematics."

Exercise LII-13* Let \( K \) be an algebraically closed field. An affine variety (see §I of Hartshorne "Algebraic Geometry") is a closed subset of \( \mathbb{A}^n := K^n \) which cannot be written as a union of two proper closed subsets, where \( K^n \) is given the topology in which sets of the form

\[
Z(T) = \{ P \in \mathbb{A}^n \mid f(P) = 0 \text{ for all } P \in T \}
\]

for \( T \subseteq K[x_1, \ldots, x_n] \), are closed (this is called the Zariski topology). Prove an affine variety is Hausdorff iff it is a singleton (this is Exercise 1.7(d) of Hartshorne).
So, I suppose algebraic geometers are strange. More seriously, as hinted at on p. 3 of Lecture 6, the purpose of a topology on $X$ is to tell you which maps out of $X$ are continuous, and in algebraic geometry there is a mismatch between the underlying topology of our spaces (which is "bad") and the topology which is implicit in the category of spaces we use (which is better).

To be more precise, even though varieties are rarely Hausdorff (and the situation for schemes is even worse!) algebraic geometers are effectively still working most of the time with "Hausdorff" spaces, since we ask for our schemes to be separated, which means the image of the diagonal

$$X \longrightarrow X \times X$$

is closed in $X \times X$ (see Exercise 4.11-11). Note this is a morphism of schemes, so this not the same as saying the underlying space of $X$ is Hausdorff (but it is the "right" notion of Hausdorff for our purposes).

So Munkres is right, conceptually, that "everyone" likes Hausdorff spaces, if that term is appropriately interpreted. However he is flat wrong about non-closed points: those are great.

**Remark** For a discussion of non-closed points in algebraic geometry see Example 2.3.4 of Hartshorne or Eisenbud, Harris "Geometry of schemes".
Solutions to selected exercises

**Ex III-10** Define \( Q^i_h := \left\{ x \in \mathbb{R}^i \mid 1-h < \| x \| \leq 1 \right\} \). This is an open subset of \( D^i \) as \( \left\{ x \mid \| x \| > 1-h \right\} \subseteq \mathbb{R}^i \) is open ( \( \| - \| : \mathbb{R}^i \rightarrow \mathbb{R} \) is continuous). Clearly \( \mathcal{S} \) is continuous and

\[
\| \mathcal{S}(x, \lambda) \| = 1 - \lambda > 1-h
\]

so \( \mathcal{S} \) factors via a continuous map \( S^{i-1} \times [0, h) \rightarrow Q^i_h \). Define \( \mathcal{S} : Q^i_h \rightarrow S^{i-1} \times [0, h) \) by

\[
\mathcal{S}(y) = \left( \frac{1}{\| y \|} y, 1-\| y \| \right).
\]

This is continuous and gives a two-sided inverse to \( S^{i-1} \times [0, h) \rightarrow Q^i_h \).

(ii) Using the notation of (i), set \( W_j := \mathcal{S}(U_j \times [0, h)) \). This is an open subset of \( Q^i_h \) and thus of \( D^i \), and \( W_1 \cap W_2 = \emptyset \) since \( (U, x [0, h)) \cap (U \times [0, h)) = \emptyset \). Moreover,

\[
W_j \cap S^{i-1} = \mathcal{S}(U_j \times [0, h)) \cap \mathcal{S}(S^{i-1} \times \{ 0 \})
\]

\[
= \mathcal{S}(U_j \times [0, h) \cap S^{i-1} \times \{ 0 \})
\]

\[
= \mathcal{S}(U_j \times \{ 0 \})
\]

\[
= U_j
\]