Lecture 10: Compactness III

Which spaces are compact? So far we know that closed and bounded subsets of \( \mathbb{R} \) (e.g. intervals \([a, b]\)) are compact (using Bolzano-Weierstrass plus Theorem L9-2). This is not a very impressive list. Let us improve it. We will begin to use “space” to mean “topological space” unless otherwise indicated.

Lemma L10-1 If \( \sim \) is an equivalence relation on a compact space \( X \) then \( X/\sim \) is compact.

Proof The quotient map \( \pi: X \rightarrow X/\sim \) is continuous so this follows from Proposition L9-3. \( \square \)

Exercise L10-1 If \( X \) and \( Y \) are homeomorphic then \( X \) is compact iff \( Y \) is compact.

So at least \( S^1 \cong [0, 1]/\sim \) is compact. If we knew \([0, 1] \times [0, 1]\) was compact we could use the same trick to show e.g. the torus is compact.

Lemma L10-2 If \( X, Y \) are compact spaces then \( X \times Y \) is compact.

Proof By Lemma L9-1 it suffices to show any open cover of \( X \times Y \) of the form \( \{U_i \times V_i\}_{i \in I} \) with \( U_i \subseteq X, V_i \subseteq Y \) open has a finite subcover. If either \( X, Y \) is empty then \( X \times Y = \emptyset \) is compact, so suppose \( X \neq \emptyset, Y \neq \emptyset \) and choose some \( x \in X \). Then \( \{x\} \times Y \subseteq X \times Y \) is covered by the \( U_i \times V_i \), and since \( \{x\} \times Y \approx Y \) is compact there is \( J_x \subseteq I \) finite such that (here we are using Ex L10-1 and Ex L9-4, to be more precise)

\[
\{x\} \times Y \subseteq \bigcup_{j \in J_x} U_j \times V_j.
\]
We may assume $x \in U_j$ for all $j \in J_x$ (otherwise $U_j \times V_j$ is superfluous and we may remove it from $J_x$) and so $x$ lies in the open set $U_x := \bigcap_{j \in J_x} U_j$. Note $U_x \times Y \subseteq \bigcup_{j \in J_x} U_j \times V_j$. Since $\{U_x\}_{x \in X}$ is an open cover of $X$ and $X$ is compact, there is a finite subcover $\{U_{x_1}, \ldots, U_{x_r}\}$. But then

$$X \times Y = \bigcup_{i=1}^r U_{x_i} \times Y \subseteq \bigcup_{i=1}^r \bigcap_{j \in J_{x_i}} U_j \times V_j.$$  

We deduce that the torus $\mathbb{T} = [0,1] \times [0,1]/\sim$ is compact. It is true that any product (even infinite ones) of compact spaces is compact, but this is much harder to prove — it is called Tychonoff’s Theorem (alas, we do not have time to prove this). Clearly by induction we deduce from the lemma that any finite product of compact spaces is compact.

**Theorem L10-3 (Heine-Borel)** A subset of $\mathbb{R}^n$ is compact iff. it is closed and bounded.

**Proof** If $Y \subseteq \mathbb{R}^n$ is compact then it is sequentially compact (Thm L9-2) and so closed and bounded by the same argument as in Bolzano-Weierstrass (see Exercise L8-2). If $Y$ is bounded then $Y \subseteq [a,b]^n$ for some $a, b$ (Exercise L8-4) and if $Y$ is additionally closed then it is a closed subspace of a compact space, hence compact (Exercise L9-5).  

**Exercise L10-2** A function $f : X \to Y$ between topological spaces is continuous iff. $f^{-1}(C)$ is closed for every closed set $C \subseteq Y$.

**Corollary L10-4** The $n$-disk $D^n \subseteq \mathbb{R}^n$ and $n$-sphere $S^n \subseteq \mathbb{R}^{n+1}$ are compact.

**Proof** We need only show these subspaces are closed. The easy way is to note that $f = \| - \| : \mathbb{R}^n \to \mathbb{R}$ is continuous, so e.g.
D^n := f^{-1}([0,1])

is closed, and similarly for S^n. □

Exercise L10-3 Recall real projective space \(\mathbb{RP}^n\) from p. 60 of Lecture 7. Show \(\mathbb{RP}^n\) is compact by exhibiting it as a quotient of \(S^n\).

Lemma L10-5 If \(X, Y\) are compact spaces then \(X \sqcup Y\) is compact.

Proof. If \(\{U_i\}_{i \in I}\) is an open cover of \(X \sqcup Y\) then (identifying \(X, Y\) with subspaces of \(X \sqcup Y\)) we have \(U_i = (U_i \cap X) \cup (U_i \cap Y)\). Choose \(J \subseteq I\) finite such that \(\{U_j \cap X\}_{j \in J}\) covers \(X\) and \(\{U_j \cap Y\}_{j \in J}\) covers \(Y\). Then \(\{U_j\}_{j \in J}\) covers \(X \sqcup Y\). □

Exercise L10-4 Let \(X\) be a space and \(Y_1, \ldots, Y_n\) compact subsets. Then \(Y_1 \cup \cdots \cup Y_n\) is compact.

Theorem L10-5 Any finite CW-complex is compact.

Proof. By definition if \(X\) is a finite CW-complex there is a sequence of spaces \(X_0, X_1, \ldots, X_n = X\) with \(X_0\) finite and discrete, and each \(X_i\) obtained from \(X_{i-1}\) via a pushout:

\[
\begin{array}{ccc}
\coprod_{a \in A} S^{i-1} & \longrightarrow & X_{i-1} \\
\downarrow & & \downarrow \\
\coprod_{a \in A} D^i & \longrightarrow & X_i = (X_{i-1} \sqcup \coprod_{a \in A} D^i)/\sim
\end{array}
\]
We prove $X$ is compact by induction: $X_0$ is clearly compact, and if $X_{i-1}$ is compact then so is $X_{i-1} \cup \bigcup_{\alpha \in \Delta} D^i$ by Corollary 40-4 and the previous lemma (it is important $\Delta$ is finite!). But then the quotient $X_i$ is also compact by Lemma 40-1, which completes the inductive step. □

That is a respectable supply of compact spaces!