THE UNIVERSITY OF MELBOURNE

Student Number

Semester 2 Assessment, 2018

School of Mathematics and Statistics

MAST30026 Metric and Hilbert Spaces

Writing time: 3 hours

Reading time: 15 minutes

This is NOT an open book exam

This paper consists of 3 pages (including this page)

Authorised Materials

- Mobile phones, smart watches and internet or communication devices are forbidden.
- Calculators, tablet devices or computers must not be used.
- No handwritten or print materials may be brought into the exam venue.

Instructions to Students

- You must NOT remove this question paper at the conclusion of the examination.
- You should attempt all questions. Marks for individual questions are shown.
- There are 6 questions with marks as shown. The total number of marks available is 110.

Instructions to Invigilators

- Students must NOT remove this question paper at the conclusion of the examination.

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Question 1 (10 marks) Let (X, d) be a metric space.

(a) State the definition of sequential compactness.

(b) Suppose that X is sequentially compact and nonempty. Given $\varepsilon > 0$ prove that there exists a finite set $x_1, \ldots, x_n \in X$ such that $\{B_\varepsilon(x_i)\}_{i=1}^n$ covers X.

You must prove (b) directly from the definition of sequential compactness.

Question 2 (20 marks) Let X be a topological space, $\Delta = \{(x, x) \in X \times X \,|\, x \in X\}$. Prove

(a) X is Hausdorff if and only if Δ is closed in $X \times X$.

(b) If X is Hausdorff, $f, g : Y \longrightarrow X$ are continuous maps and $A \subseteq Y$ is dense, then $f = g$ if and only if $f(a) = g(a)$ for all $a \in A$.

Question 3 (20 marks) Let X be locally compact Hausdorff and Y, Z topological spaces. Let

$$\pi_Y : Y \times Z \longrightarrow Y\,, \qquad \pi_Z : Y \times Z \longrightarrow Z$$

be the projection maps. Prove that the function

$$\begin{aligned} \mathrm{Cts}(X, Y \times Z) &\longrightarrow \mathrm{Cts}(X, Y) \times \mathrm{Cts}(X, Z) \\ f &\longmapsto (\pi_Y \circ f, \pi_Z \circ f) \end{aligned}$$

is a homeomorphism, with respect to the compact-open topology. You may assume the universal property of the product, and the adjunction property for the compact-open topology (including continuity of evaluation maps).

Question 4 (20 marks) Let $(V, \|-\|)$ be a normed space over a field of scalars $\mathbb{F}$ (which recall denotes either $\mathbb{R}$ or $\mathbb{C}$).

(a) Prove that $\|-\| : V \longrightarrow \mathbb{F}$ is uniformly continuous.

Prove that V is a topological vector space by proving

(b) The addition $V \times V \longrightarrow V$ is continuous.

(c) The scalar multiplication $\mathbb{F} \times V \longrightarrow V$ is continuous.

You may prove continuity using either the product topology or the product metric.

Question 5 (20 marks) Let $(V, \|-\|)$ be a normed space over a field of scalars $\mathbb{F}$ and let $V^\vee$ denote the space of continuous linear maps $V \longrightarrow \mathbb{F}$ with the operator norm. You may assume that this is a normed space. Prove that this space is *complete*, as follows:

(a) Given a Cauchy sequence $(T_n)_{n=0}^\infty$ in $V^\vee$ with respect to the operator norm, construct a candidate limit T as a function $T : V \longrightarrow \mathbb{F}$.

(b) Prove that your candidate T is linear.

(c) Prove that your candidate T is bounded.

(d) Prove that $T_n \longrightarrow T$ in the operator norm as $n \longrightarrow \infty$.

Question 6 (20 marks) Let $(H, \langle -, - \rangle)$ be a Hilbert space over $\mathbb{C}$.

(a) State the Cauchy-Schwartz inequality.

(b) Prove that for any $h \in H$ the function $\langle -, h \rangle : H \longrightarrow \mathbb{C}$ is continuous.

(c) Prove that if $\{u_i\}_{i \in I}$ is a set of vectors in H which span a vector subspace $U \subseteq H$ with the property that U is dense in H, then $h = 0$ if and only if $\langle u_i, h \rangle = 0$ for all $i \in I$.

(d) Given that $\{e^{in\theta}\}_{n \in \mathbb{Z}}$ span a dense subspace of $H = L^2(S^1, \mathbb{C})$ prove that for every $f \in H$

$$f = \lim_{N \longrightarrow \infty} \sum_{n=-N}^{N} \frac{1}{2\pi} \langle f, e^{in\theta} \rangle e^{in\theta} .$$

You may assume that the series on the right hand side converges.

End of Exam—Total Available Marks = 110