Exam solutions (MAST30026 S2 2018) Total 110

⌜This is Lemma L9-1⌟

Q1 (10) (a) A metric space (X,d) is sequentially compact if every sequence in X has a convergent subsequence. — (2)

(b) Given $\varepsilon > 0$ suppose no such finite cover of ε-balls existed. (2) Choose some $x_1 \in X$. (2) Then $B_\varepsilon(x_1) \neq X$ so we may choose $x_2 \in X \setminus B_\varepsilon(x_1)$. Since $B_\varepsilon(x_1) \cup B_\varepsilon(x_2) \neq X$ we may choose $x_3 \notin X \setminus B_\varepsilon(x_1) \cup B_\varepsilon(x_2)$ and in this way construct $(x_n)_{n=0}^{\infty}$ with $d(x_n, x_m) \geqslant \varepsilon$ whenever $n \neq m$.

(2) construction of a sequence (even w.o observing $d \geqslant \varepsilon$)

By hypothesis $(x_n)_{n=0}$ contains a convergent subsequence, but this subsequence clearly cannot be Cauchy and so we have a contradiction.

complete argument (2)

⌜Q2a is Ex. L11-11 (i), from assignment #3⌟

Q2 (20) (a) Suppose X is Hausdorff and $(x,y) \notin \Delta$, i.e. $x \neq y$. Let U, V be open disjoint sets with $x \in U, y \in V$. Then $(x,y) \in U \times V$ and $U \times V \subseteq \Delta^c$, so Δ^c is open and hence Δ is closed. — (5)

If Δ is closed and $x \neq y$ then $(x,y) \in \Delta^c$ has an open neighborhood of the form $U \times V$ and so $x \in U, y \in V$ and $U \times V \cap \Delta = \phi$ implies U, V are disjoint so X is Hausdorff. — (5)

⌜This is Lemma L17-2⌟

(b) Consider the continuous map $(\Delta(x) = (x,x))$

$$X \xrightarrow{\Delta} X \times X \xrightarrow{f \times g} Y \times Y$$ (2)

(2)

since Y is Hausdorff the diagonal $\Delta = \{(y,y) \mid y \in Y\} \subseteq Y \times Y$ is closed, and its preimage under the above map $\{x \in X \mid f(x) = g(x)\}$ is therefore closed in X. (2) Hence if $A \subseteq X$ is dense and $f|_A = g|_A$ then $A \subseteq \{x \mid f(x) = g(x)\}$ and therefore $\{x \mid f(x) = g(x)\} = X$. $\square$

(2) (2)

「This was Ex. L12-14」

$\boxed{Q3}$ (20) For X locally compact Hausdorff the evaluation map

$$X \times \mathrm{Cts}(X,Y) \xrightarrow{ev_{X,Y}} Y$$

(10) for proving the function is continuous

(10) for proving it is a homeomorphism

is continuous, and hence if $f : Y \to Z$ is continuous so is

$$X \times \mathrm{Cts}(X,Y) \xrightarrow{ev_{X,Y}} Y \xrightarrow{f} Z$$

and so by the adjunction property

$$\mathrm{Cts}(X,Y) \xrightarrow{f \circ (-)} \mathrm{Cts}(X,Z)$$

(5)

is continuous. It therefore follows from the universal property of the product that the given map in the question is continuous. (5) It suffices to produce a continuous inverse, which corresponds under the adjunction property to

$$\begin{array}{c} X \times \mathrm{Cts}(X,Y) \times \mathrm{Cts}(X,Z) \\ \big\downarrow \Delta \times 1 \\ X \times X \times \mathrm{Cts}(X,Y) \times \mathrm{Cts}(X,Z) \\ \cong \\ X \times \mathrm{Cts}(X,Y) \times X \times \mathrm{Cts}(X,Z) \xrightarrow{ev_{X,Y} \times ev_{X,Z}} Y \times Z. \end{array}$$

This map is continuous by construction, and sends $(x, \pi_Y \circ f, \pi_Z \circ f)$ to $f(x)$, so it corresponds under the Ψ map of lectures to an inverse for $\mathrm{Cts}(X, Y \times Z) \longrightarrow \mathrm{Cts}(X,Y) \times \mathrm{Cts}(X,Z)$.

$\boxed{Q4}$ (a) The reverse triangle inequality says $\big|\, \|v\| - \|w\| \,\big| \leq \|v-w\|$
(20)
so given $\varepsilon > 0$ if $d(v,w) < \varepsilon$ then $d(\|v\|, \|w\|) < \varepsilon$ hence $\|-\|$ is uniformly continuous. ⌜This is Lemma L18-3⌟ — 5

(b),(c) We prove first that $+ : V \times V \longrightarrow V$ is continuous at (v_0, w_0), by calculating ⌜(b),(c) are Ex. L18-10⌟

$$\| +(v_1,w_1) - +(v_0,w_0) \| = \| v_1 - v_0 + w_1 - w_0 \|$$
$$\leq \|v_1 - v_0\| + \|w_1 - w_0\|$$

with respect to the product metric on $V \times V$ this is

$$= d_{V\times V}\big((v_1,w_1), (v_0,w_0)\big)$$ — 5

so continuity is clear. Similarly for scalar multiplication

$$\begin{aligned}
\| \lambda_1 v_1 - \lambda_0 v_0 \| &= \| \lambda_1 v_1 - \lambda_0 v_1 + \lambda_0 v_1 - \lambda_0 v_0 \| \\
&= \| (\lambda_1 - \lambda_0) v_1 + \lambda_0 (v_1 - v_0) \| \\
&\leq \| (\lambda_1 - \lambda_0) v_1 \| + \| \lambda_0 (v_1 - v_0) \| \\
&= |\lambda_1 - \lambda_0| \, \|v_1\| + |\lambda_0| \, \|v_1 - v_0\| \\
&= |\lambda_1 - \lambda_0| \, \|v_1 - v_0 + v_0\| + |\lambda_0| \, \|v_1 - v_0\| \\
&\leq |\lambda_1 - \lambda_0| \, \|v_1 - v_0\| + |\lambda_1 - \lambda_0| \, \|v_0\| \\
&\quad + |\lambda_0| \, \|v_1 - v_0\|
\end{aligned}$$ — 5

Given $\varepsilon > 0$ choose $\delta \leq \min\left\{ \sqrt{\frac{\varepsilon}{3}}, \frac{\varepsilon}{3\|v_0\|}, \frac{\varepsilon}{3|\lambda_0|} \right\}$

Then if $d_{\mathbb{F}\times V}\big((\lambda_0, v_0), (\lambda_1, v_1)\big) < \delta$ we have

$$|\lambda_0 - \lambda_1| + \|v_0 - v_1\| < \delta$$

and hence $|\lambda_0 - \lambda_1| < \delta$, $\|v_0 - v_1\| < \delta$ and so

$$\|\lambda_1 v_1 - \lambda_0 v_0\| \leq |\lambda_1 - \lambda_0| \, \|v_1 - v_0\| + |\lambda_1 - \lambda_0| \, \|v_0\| + |\lambda_0| \, \|v_1 - v_0\|$$

$$< \delta^2 + \delta \|v_0\| + |\lambda_0| \delta$$

$$\leq \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon$$

5

as required.

This is Ex. L19-3

$\boxed{Q5}$ (20)

(a) Let $(T_n)_{n=0}^{\infty}$ be given. We claim for $v \in V$ that $(T_n(v))_{n=0}^{\infty}$ is Cauchy in $\mathbb{F}$. Let $\varepsilon > 0$ be given and choose N s.t. for all $n \geqslant N$ $\|T_m - T_n\| < \varepsilon$. Then

$$\|T_m(v) - T_n(v)\| = \|(T_m - T_n)(v)\| \leq \|T_m - T_n\| \cdot \|v\| \leq \varepsilon \|v\|$$

It follows that $(T_n(v))_{n=0}^{\infty}$ is Cauchy and thus converges in F. We set

$$T(v) := \lim_{n \to \infty} T_n(v).$$

5

(b) $T(v+w) = \lim_{n\to\infty} T_n(v+w)$

$$= \lim_{n\to\infty} \left(T_n(v) + T_n(w) \right)$$

$$= \lim_{n\to\infty} T_n(v) + \lim_{n\to\infty} T_n(w)$$

$$= T(v) + T(w)$$

$$T(\lambda v) = \lim_{n\to\infty} T_n(\lambda v)$$

$$= \lim_{n\to\infty} (\lambda T_n(v))$$

$$= \lambda \lim_{n\to\infty} T_n(v)$$

$$= \lambda T(v).$$

5

(c) We need to estimate

$$|T(v)| = |\lim_{n\to\infty} T_n(v)|$$

$$= \lim_{n\to\infty} |T_n(v)|$$

Now we would like to say this is less than $\lim_{n\to\infty} \|T_n\| \cdot \|v\|$ but we must first argue that $\lim_{n\to\infty} \|T_n\|$ exists. But $\|-\| : V \longrightarrow \mathbb{F}$ is uniformly continuous so $(\|T_n\|)_{n=0}^{\infty}$ is Cauchy and hence converges. So

$$\leq \lim_{n\to\infty} \|T_n\| \cdot \|v\|$$

$$= \left(\lim_{n\to\infty} \|T_n\|\right) \cdot \|v\|.$$

This shows T is bounded and moreover $\|T\| \leq \lim_{n\to\infty} \|T_n\|$.

5

(d) We need to show $\|T - T_n\| \to 0$ as $n \to \infty$. Note that with $m, n \in \mathbb{N}$ and $v \in V$ with $\|v\| \leq 1$

$$|T_m(v) - T_n(v)| \leq \|T_m - T_n\|$$

Let N be sufficiently large that this is $< \varepsilon$. Take $n \to \infty$ to obtain

$$|T_m(v) - T(v)| \leq \varepsilon \qquad m \geq N.$$

Hence

$$\|T_m - T\| = \sup\{ |T_m(v) - T(v)| \mid \|v\| \leq 1\} \leq \varepsilon.$$ 5

as claimed.

Q6 (a) $|\langle a, b\rangle| \leq \|a\| \cdot \|b\|$ 5

"Material covered in lectures"

(b) Let $f(a) = \langle a, h\rangle$. Then if $h = 0$ $f \equiv 0$ is continuous. If $h \neq 0$,

$$|f(a) - f(b)| = |\langle a, h\rangle - \langle b, h\rangle|$$
$$= |\langle a - b, h\rangle|$$
$$\leq \|a - b\| \cdot \|h\|.$$

To prove f is continuous at b, let $\varepsilon > 0$ be given and set $\delta = \varepsilon / \|h\|$. Then $\|a - b\| < \delta$ implies $|f(a) - f(b)| < \delta\|h\| = \varepsilon$. 5

(c) Suppose $\langle u_i, f\rangle = 0$ for all $i \in I$. Since $\langle -, h\rangle$ is linear and continuous $\{h\}^{\perp}$ is a closed vector subspace. If it contains $\{u_i\}_{i\in I}$ then it contains U and thus $\overline{U} = H$ so $h = 0$. — 5

(d) Set $U = \operatorname{span}_{\mathbb{C}}(\{e^{in\theta}\}_{n\in\mathbb{Z}})$ so $\overline{U} = H$. We set $u_n = \frac{1}{\sqrt{2\pi}} e^{in\theta}$ and

$$f' = \lim_{N\to\infty} \sum_{n=-N}^{N} \langle f, u_n\rangle u_n$$

Then

$$\begin{aligned}
\langle u_m, f - f'\rangle &= \langle u_m, f\rangle - \langle u_m, f'\rangle \\
&= \langle u_m, f\rangle - \lim_{N\to\infty} \sum_{n=-N}^{N} \langle u_m, \langle f, u_n\rangle u_n\rangle \\
&= \langle u_m, f\rangle - \lim_{N\to\infty} \sum_{n=-N}^{N} \langle u_n, f\rangle \langle u_m, u_n\rangle \\
&= \langle u_m, f\rangle - \langle u_m, f\rangle = 0
\end{aligned}$$

Hence $f = f'$. — 5