University of Melbourne
MAST30026 Metric & Hilbert Spaces

ASSIGNMENT 3

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Contents

1 Exercise L16-3.5 ($\bar{A}$ *is a subalgebra*)

Lemma 1.1. *Let X be compact and (Y, d_Y) a metric space. Given a subset $A \subseteq \mathrm{Cts}(X, Y)$, the following conditions on $f \in \mathrm{Cts}(X, Y)$ are equivalent:*

1. *$f \in \bar{A}$;*
2. *there is a sequence $(a_n)_{n=0}^{\infty}$ in A converging uniformly to f;*
3. *f may be uniformly approximated by elements of A, that is, given $\varepsilon > 0$ there exists $a \in A$ such that $|f(x) - a(x)| < \varepsilon$ for all $x \in X$.*

Proof. From the notes (*Exercise L16-1*). □

Proposition 1.1. *Let X be compact Hausdorff and $A \subseteq \mathrm{Cts}(X, \mathbb{R})$ a subalgebra. Then $\bar{A} \subseteq \mathrm{Cts}(X, \mathbb{R})$ is also a subalgebra.*

Proof. Since X is compact and $\mathbb{R}$ is a metric space, we can suitably define the d_∞ metric on $\mathrm{Cts}(X, \mathbb{R})$.

Let $f, g \in \bar{A}$. By Lemma 1.1, there exist sequences $(f_n)_{n=0}^{\infty}, (g_n)_{n=0}^{\infty}$ in A converging uniformly to f, g respectively. Equivalently, the sequences converge to f, g with respect to the d_∞ metric on $\mathrm{Cts}(X, \mathbb{R})$.

Since A is a subalgebra of $\mathrm{Cts}(X, \mathbb{R})$, it is closed under the operations of addition, multiplication, and scalar multiplication. Thus, given $f_n, g_n \in A$ above and $\lambda \in \mathbb{R}$, the sequences

$$(f_n + g_n)_{n=0}^{\infty}, \quad (f_n g_n)_{n=0}^{\infty}, \quad (\lambda f_n)_{n=0}^{\infty} \tag{1.1}$$

are all sequences in $A \subseteq \mathrm{Cts}(X, \mathbb{R})$.

Note that X being compact is a stronger conditions than local compactness. Thus by *Lemma L16-6* of the lecture notes, $\mathrm{Cts}(X, \mathbb{R})$ is a topological $\mathbb{R}$-algebra: the operations of addition, multiplication, and scalar multiplication are continuous. Thus, these operations commute with limits, and we have

$$\begin{aligned}
f + g &= \lim_{n\to\infty} f_n + \lim_{n\to\infty} g_n \\
&= \lim_{n\to\infty} (f_n + g_n) \in \bar{A}, &(1.2)\\
f \cdot g &= \left(\lim_{n\to\infty} f_n\right) \cdot \left(\lim_{n\to\infty} g_n\right) \\
&= \lim_{n\to\infty} f_n g_n \in \bar{A}, &(1.3)\\
\lambda f &= \lambda \lim_{n\to\infty} f_n \\
&= \lim_{n\to\infty} \lambda f \in \bar{A}, &(1.4)
\end{aligned}$$

where we have concluded Equation 1.2, Equation 1.3, Equation 1.4 from combinbing Equation 1.1 with Lemma 1.1. Thus $\bar{A}$ is closed under addition, multiplication, and scalar multiplication.

Since A is a subalgebra, we have $1 \in A$ and so $1 \in \bar{A}$.[1] Thus we ultimately have that $\bar{A}$ is a subalgebra. □

[1] Using Lemma 1.1, we can construct the constant sequence $(1)_{n=0}^{\infty}$ to see this.

2 Exercise L16-14 (*Stone-Weierstrass for locally compact spaces*)

Lemma 2.1. *Let X be locally compact Hausdorff. Let $\hat{X}$ be the one-point compactification of X. Let*

$$\hat{C} := \left\{ f \in \operatorname{Cts}\left(\hat{X}, \mathbb{R}\right) \mid f(\infty) = 0 \right\}. \tag{2.1}$$

Then $\hat{C}$ is an (non-unital) algebra and is isometrically isomorphic to $\operatorname{Cts}_0(X, \mathbb{R})$, the continuous functions on X that vanish at infinity.

Proof. Clearly $\hat{C}$ is a (non-unital) subalgebara, since any sum, product, or scalar product of functions in it are still zero at ∞, so $\hat{C}$ is closed under the operations Define

$$\psi : \operatorname{Cts}_0(X, \mathbb{R}) \to \hat{C}, \quad \psi(f)|_X = f, \quad \psi(f)(\infty) = 0. \tag{2.2}$$

The map ψ is linear, and we see clearly that $\|f\|_\infty = \|\psi(f)\|_\infty$. Clearly the kernel is trivial, so ψ is injective. Furthemore, given $h \in \hat{C}$, we simply have $h|_X \in \operatorname{Cts}_0(X, \mathbb{R})$, so ψ is surjective. ☐

Lemma 2.2. *Let X be a compact Hausdorff space and $A \subseteq \operatorname{Cts}(X, \mathbb{R})$ a subalgebra that separates points, and is such that $\forall f \in A$, $f(\xi) = 0$ for some $\xi \in X$. Then*

$$\bar{A} = \left\{ f \in \operatorname{Cts}(X, \mathbb{R}) \mid f(\xi) = 0 \right\}. \tag{2.3}$$

Proof. The inclusion $\bar{A} \subseteq \left\{ f \in \operatorname{Cts}(X, \mathbb{R}) \mid f(\xi) = 0 \right\}$ essentially follows from the same arguments given in Proposition 1.1.

For the opposite inclusion, first define

$$A' := \left\{ f + a \mid f \in A, a \in \mathbb{R} \text{ a constant function} \right\}. \tag{2.4}$$

Since A' inherits the algebraic structure of A, it is clear to see that A' is a non-unital subalgebra of $\operatorname{Cts}(X, \mathbb{R})$. Since we may take $a = 0$, we see that $A \subset A'$, and so A' separates points. Furthermore, since we may take $f = 0 \in A$, we see that A' contains 1 and so it is in fact a (unital) subalgebra. Therefore, by the **Stone-Weierstrass Theorem**, we have

$$\bar{A}' = \operatorname{Cts}(X, \mathbb{R}). \tag{2.5}$$

Now let $f \in \operatorname{Cts}(X, \mathbb{R})$ be such that $f(\xi) = 0$. By Lemma 1.1, there exists a sequence $(f'_n)_{n=0}^\infty$ in A' converging to f with respect to the d_∞ metric. By definition we may write

$$f'_n = f_n + a_n, \quad f_n \in A, a_n \in \mathbb{R}. \tag{2.6}$$

Since $f'_n(\xi) \to 0$ as $n \to \infty$, and $f_n(\xi) = 0$ by definition, we must have $a_n \to 0$ in the limit. Thus $f'_n - f_n \to 0$, and so the two sequences are equivalent Cauchy sequences. Thus $f_n \to f$, and so $f \in \bar{A}$, as was to be shown. ☐

Proposition 2.1. *Let X be locally compact Hausdorff and A a non-unital subalgebra of $\operatorname{Cts}_0(X, \mathbb{R})$ that separates points and is such that $\forall x \in X$ there exists some $f \in A$ such that $f(x) \neq 0$. Then $\bar{A} = \operatorname{Cts}_0(X, \mathbb{R})$.*

Proof. By Lemma 2.1, we may view A as a subalgebra $A \cong \hat{A}$ in $\mathrm{Cts}\left(\hat{X}, \mathbb{R}\right)$ which separates points in $\hat{X}$ and is zero at ∞. Then by Lemma 2.2, we have $\bar{\hat{A}} = \hat{C}$. Thus by going back through the isometric isomorphism we have that $\bar{A} = \mathrm{Cts}_0\left(X, \mathbb{R}\right)$. □

3 Exercise L17-2 ($\exp\cos\theta$)

Proposition 3.1. *The function* $\exp(\cos\theta)$ *is **not** in the linear span of the set* $\{\cos(n\theta), \sin(n\theta)\}_{n\geq 1}\cup\{1\}$.

Proof. By definition, the span of a set consists of all the **finite** linear combinations of elements of the set. By definition of the exponential, we may write

$$\begin{aligned}\exp(\cos\theta) &= \sum_{n=0}^{\infty}\frac{\cos^n\theta}{n!}\\ &= \sum_{n \text{ even}}\frac{1}{2^n}\binom{n}{n/2} + \frac{2}{2^n}\sum_{k=0}^{n/2-1}\binom{n}{k}\cos((n-2k)\theta) + \sum_{n \text{ odd}}\frac{2}{2^n}\sum_{k=0}^{\frac{n-1}{2}}\binom{n}{k}\cos((n-2k)\theta),\end{aligned} \tag{3.1}$$

where we have used the power reduction laws. Crucially, we see that the terms in the series are non-zero for arbitrary n. In particular, by inspecting the terms above for $k=0$ we have a contribution of at least[2]

$$\sum_{n=0}^{\infty}\frac{2}{2^n}\cos(n\theta). \tag{3.2}$$

So there is **no** finite $\mathbb{R}$-linear combination

$$S = a_0 + \sum_{n=1}^{r} a_n\cos(n\theta) + b_n\sin(n\theta \tag{3.3}$$

such that $\exp(\cos\theta) - S = 0$, since there are always non-zero coefficients at least of the form

$$\frac{2}{2^n}, \tag{3.4}$$

with index $n > r$, given any $r\in\mathbb{N}$. ☐

[2] Note that all the coefficients are positive.

4 Exercise L19-4 (*Bounded dual*)

Proposition 4.1. *Let* $(V, \|\cdot\|_V)$ *and* $(W, \|\cdot\|_W)$ *be normed spaces and* $T : V \to W$ *a bounded linear operator. Then*

$$\begin{aligned} T^\vee &: W^\vee \to V^\vee \\ T^\vee(g) &= g \circ T \end{aligned} \tag{4.1}$$

is a bounded linear operator with $\|T^\vee\| \leq \|T\|$*. Moreover,* $(\cdot)^\vee$ *is a functor, that is,*

$$(\mathrm{id}_V)^\vee = \mathrm{id}_{V^\vee}, \tag{4.2}$$

and, given a bounded linear operator $S : W \to U$

$$(S \circ T)^\vee = T^\vee \circ S^\vee. \tag{4.3}$$

Proof. We first show that $T^\vee$ is linear. Given $f, g \in W^\vee$, we have, for all $v \in V$

$$\begin{aligned} T^\vee(f+g)(v) &= (f+g) \circ T(v) \\ &= f(T(v)) + g(T(v)) && \text{(by definition of addition of maps)} \\ &= f \circ T(v) + g \circ T(v) \\ &= T^\vee(f)(v) + T^\vee(g)(v) \\ \implies T^\vee(f+g) &= T^\vee(f) + T^\vee(g). \end{aligned} \tag{4.4}$$

Similarly, for $\lambda \in \mathbb{F}$, we have for all $v \in V$

$$\begin{aligned} T^\vee(\lambda g)(v) &= (\lambda g) \circ T(v) \\ &= \lambda \cdot f(T(v)) && \text{(by definition of scalar multiplication of maps)} \\ &= \lambda T^\vee(g)(v) \\ \implies T^\vee(\lambda g) &= \lambda T^\vee(g). \end{aligned} \tag{4.5}$$

We now seek to show $T^\vee$ is bounded, i.e. we wish to find an $M \geq 0$ such that $\|T^\vee(g)\|_{V^\vee} \leq M \|g\|_{W^\vee}$. Since $g \in W^\vee$ is a continuous linear functional, it is bounded and permits an operator norm $\|g\|_{W^\vee} = \|g\|$. We thus have, for all $w \in W$.

$$\|g(w)\|_{\mathbb{F}} \leq \|g\| \, \|w\|_W. \tag{4.6}$$

In particular we have, for all $v \in V$

$$\begin{aligned} \|(g \circ T)(v)\|_{\mathbb{F}} &= \|g(Tv)\|_{\mathbb{F}} \\ &\leq \|g\| \, \|Tv\|_W \\ &\leq \|g\| \, \|T\| \, \|v\|_V. \end{aligned} \tag{4.7}$$

So for $\|v\|_V \neq 0$ we have

$$\frac{\|(g \circ T)(v)\|_{\mathbb{F}}}{\|v\|_V} \leq \|g\| \, \|T\|. \tag{4.8}$$

This gives us an upper bound for the left-hand side. The operator norm is defined as the supremum of the left-hand side over all suitable v. Since the supremum is the **least** upper bound, we therefore have

$$\|T^{\vee}(g)\|_{V^{\vee}} = \|g \circ T\|_{V^{\vee}} \leq \|T\| \, \|g\|_{W^{\vee}} \, . \tag{4.9}$$

We have thus shown that $T^{\vee}$ is bounded, with $M = \|T\|$. We now have

$$\sup \left\{ \frac{\|T^{\vee}(g)\|_{V^{\vee}}}{\|g\|_{W^{\vee}}} \mid g \neq 0 \right\} \leq \sup \left\{ \|T\| \, \middle| \, g \neq 0 \right\} = \|T\| \, , \tag{4.10}$$

since $\|T\|$ is independent of g. We have thus shown that

$$\|T^{\vee}\| \leq \|T\| \, , \tag{4.11}$$

as required.

We now show that $(\cdot)^{\vee}$ is a functor. Observe

$$\begin{aligned} (\mathrm{id}_V)^{\vee} : V &\longrightarrow V, \\ g &\longmapsto g \circ \mathrm{id}_V = g, \end{aligned} \tag{4.12}$$

since for all $v \in V$ we have $(g \circ \mathrm{id}_V)(v) = g(\mathrm{id}_V \, v) = gv$. Now suppose we have $S : W \to U$ a bounded linear operator (hence also continuous). Then we have

$$\begin{aligned} (S \circ T)^{\vee} : U^{\vee} &\to W^{\vee} \\ g \mapsto g \circ S \circ T &= (g \circ S) \circ T \\ &= S^{\vee}(g) \circ T \\ &= T^{\vee}\left(S^{\vee}(g)\right) \\ &= \left(T^{\vee} \circ S^{\vee}\right)(g). \end{aligned} \tag{4.13}$$

Thus $(S \circ T)^{\vee} = T^{\vee} \circ S^{\vee}$ as required. $\square$