(3) Exercise L17-2
Prove $\exp\{\cos(\theta)\} \in \mathrm{Cts}(S^1, \mathbb{R})$ is not in the linear span of $\{\cos(n\theta), \sin(n\theta)\}_{n\geq 1} \cup \{1\}$.

Answer:
Suppose that $\exp\{\cos(\theta)\}$ is in the linear span, i.e. there exist some coefficients $a_0, \{a_n, b_n\}_{n=1}^{N}$ for some $N < \infty$ such that

$$\exp\{\cos(\theta)\} = a_0 + \sum_{n=1}^{N} a_n \cos(n\theta) + \sum_{n=1}^{N} b_n \sin(n\theta).$$

First, we argue that all the b_n are zero. We know that cos is even, and sin is odd, so it follows that

$$\sum_{n=1}^{N} b_n \sin(n\theta) = 0.$$

Since $\sin(n\theta)$ is linearly independent, then we know that all b_n must be zero.
Since $\exp\{\cos(\theta)\}$ was supposed to be in the linear span, we know that there exist some nonzero a_n such that

$$\exp\{\cos(\theta)\} - a_0 - \sum_{n=1}^{N} a_n \cos(n\theta) = 0$$

The goal is to differentiate this equation. We can easily see that

$$\begin{aligned}\frac{\partial^{(2k)}}{\partial\theta^{(2k)}} \cos(n\theta) &= n^{2k}(-1)^k \cos(n\theta), \quad k = 1, 2, \ldots \\ \frac{\partial^{(2k-1)}}{\partial\theta^{(2k-1)}} \cos(n\theta) &= (-1)^k n^{2k-1} \sin(n\theta), \quad k = 1, 2, \ldots.\end{aligned}$$

Now, we can see that when $\theta = 0$, $\sin(n\theta) = 0$, so we can ignore all odd derivatives. On the other hand, when $\theta = 0$, $\cos(n\theta) = 1$. Setting $D(k) = \frac{\partial^k}{\partial\theta^k} \exp\{\cos\theta\}|_{\theta=0}$, we only need to consider equations of the form

$$D(k) - \sum_{n=1}^{N} a_n (-1)^k n^{2k}, \quad k = 1, 2, \ldots$$

Moreover, we know that when $\theta = \pi/2$,

$$\begin{aligned}\frac{\partial^2}{\partial\theta^2} \exp\{\cos\theta\}|_{\theta=\pi/2} &= \frac{\partial}{\partial\theta}(-\sin\theta)\exp\{\cos\theta\}|_{\theta=\pi/2} \\ &= (\sin^2\theta - \cos\theta)\exp\{\cos\theta\}|_{\theta=\pi/2} \\ &= e > 0.\end{aligned}$$

So, we get the following system of equations:

$$\begin{bmatrix} e & 0 & 0 & 0 & \cdots & 0 \\ D(2) & 1 & 2^2 & 3^2 & \cdots & N^2 \\ -D(4) & 1 & (2^2)^2 & (3^2)^2 & \cdots & (N^2)^2 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -D(2N) & 1 & (2^2)^N & (3^2)^N & \cdots & (N^2)^N \end{bmatrix} \begin{bmatrix} 1 \\ a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

The bottom diagonal can be coerced into a Vandermonde matrix by taking out a diagonal:

$$\begin{bmatrix} 1 & 2^2 & 3^2 & \cdots & N^2 \\ 1 & (2^2)^2 & (3^2)^2 & \cdots & (N^2)^2 \\ 1 & (2^2)^4 & (3^2)^4 & \cdots & (N^2)^4 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & (2^2)^N & (3^2)^N & \cdots & (N^2)^N \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 2^2 & 3^2 & \cdots & N^2 \\ 1 & (2^2)^2 & (3^2)^2 & \cdots & (N^2)^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & (2^2)^{N-2} & (3^2)^{N-2} & \cdots & (N^2)^{N-2} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 2^2 & 0 & \cdots & 0 \\ 0 & 0 & 3^2 & \cdots & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & N^2 \end{bmatrix}$$

The determinant is clearly nonzero. Moreover, for the larger matrix $e > 0$, so the determinant of the whole matrix is nonzero, and it is invertible. We find that $1 = 0$. Contradiction.