Exercise L14-1

Throughout this question, we will be using the d_∞ metric on $\mathbb{R}^n$.

We'll first show that the function $f\colon \mathbb{R}^n \to \mathbb{R}^n$ given by $f(\mathbf{v}) = \mathbf{v} - A\mathbf{v} + \mathbf{w}$ (where $\mathbf{w}$ is a fixed vector and A is as given in the question) is a contraction. To do this, we will be using the hypothesis given in the problem statement, i.e. that there exists $\lambda \in (0,1)$ with

$$\sum_{j=1}^{n} |\delta_{ij} - A_{ij}| \le \lambda \qquad \text{for each } 1 \le i \le n \tag{1}$$

We claim that f is a λ-contraction. Indeed, consider any two vectors $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$. We have

$$d_\infty(\mathbf{x}, \mathbf{y}) = \max\{|x_i - y_i|\}_{i=1}^n$$

and

$$\begin{aligned}
d_\infty(f(\mathbf{x}), f(\mathbf{y})) &= d_\infty(\mathbf{x} - A\mathbf{x} + \mathbf{w}, \mathbf{y} - A\mathbf{y} + \mathbf{w}) \\
&= \max\{|\pi_i((\mathbf{x} - A\mathbf{x} + \mathbf{w}) - (\mathbf{y} - A\mathbf{y} + \mathbf{w}))|\})_{i=1}^n \\
&= \max\{|\pi_i(\mathbf{x} - \mathbf{y} - A\mathbf{x} + A\mathbf{y})|\})_{i=1}^n \\
&= \max\{|\pi_i((I_n - A)(\mathbf{x} - \mathbf{y}))|\})_{i=1}^n
\end{aligned}$$

where $\pi_i\colon \mathbb{R}^n \to \mathbb{R}$, $1 \le i \le n$, denotes the projection and I_n is the $n \times n$ identity matrix. To calculate $\pi_i((I_n - A)(\mathbf{x} - \mathbf{y}))$ (for $1 \le i \le n$) we need to take the dot product of the ith row of $I_n - A$ with $\mathbf{x} - \mathbf{y}$, i.e.

$$\pi_i((I_n - A)(\mathbf{x} - \mathbf{y})) = \sum_{j=1}^{n} (\delta_{ij} - A_{ij})(x_j - y_j)$$

And now we have, for each $1 \le i \le n$,

$$\begin{aligned}
|\pi_i((I_n - A)(\mathbf{x} - \mathbf{y}))| &= \left| \sum_{j=1}^{n} (\delta_{ij} - A_{ij})(x_j - y_j) \right| \\
&\le \sum_{j=1}^{n} |\delta_{ij} - A_{ij}| \cdot |x_j - y_j| && \text{(by the triangle inequality)} \\
&\le \sum_{j=1}^{n} |\delta_{ij} - A_{ij}| \cdot d_\infty(\mathbf{x}, \mathbf{y}) && \text{(since } d_\infty(\mathbf{x}, \mathbf{y}) = \max\{|x_j - y_j|\}_{j=1}^n) \\
&= d_\infty(\mathbf{x}, \mathbf{y}) \sum_{j=1}^{n} |\delta_{ij} - A_{ij}| \\
&\le \lambda\, d_\infty(\mathbf{x}, \mathbf{y}) && \text{(by (1))}
\end{aligned}$$

and so we have that $|\pi_i((I_n - A)(\mathbf{x} - \mathbf{y}))| \le \lambda\, d_\infty(\mathbf{x}, \mathbf{y})$ for every $1 \le i \le n$. This implies

$$d_\infty(f(\mathbf{x}), f(\mathbf{y})) = \max\{|\pi_i((I_n - A)(\mathbf{x} - \mathbf{y}))|\})_{i=1}^n \le \lambda\, d_\infty(\mathbf{x}, \mathbf{y})$$

so that $d_\infty(f(\mathbf{x}), f(\mathbf{y})) \le \lambda\, d_\infty(\mathbf{x}, \mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}$ and hence f is a λ-contraction, as claimed.

From here, observe that since $(\mathbb{R}^n, d_\infty)$ is a complete metric space and $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is a contraction, we may use the Banach fixed point theorem to conclude that f has a unique fixed point. Thus there is a unique value of $\mathbf{v} \in \mathbb{R}^n$ such that $f(\mathbf{v}) = \mathbf{v}$, i.e. $\mathbf{v} - A\mathbf{v} + \mathbf{w} = \mathbf{v}$. This implies that $A\mathbf{v} = \mathbf{w}$ has a unique solution $\mathbf{v}$, as required.

Exercise L14-2

Let (X,d) be a compact metric space, and let $\lambda \in (0,1)$. To show that the map fix : $\mathrm{Cts}_\lambda(X,X) \to X$ is continuous, it suffices (by Lemma L6-4) to show that for all $f \in \mathrm{Cts}_\lambda(X,X)$, given any $\epsilon > 0$ there exists $\delta > 0$ such that

$$d_\infty(f,g) < \delta \implies d(\mathrm{fix}(f), \mathrm{fix}(g)) < \epsilon \qquad \forall g \in \mathrm{Cts}_\lambda(X,X).$$

Here d_∞ is defined as usual, i.e.

$$d_\infty(f,g) = \sup\{d(f(x), g(x)) \mid x \in X\}$$

(which is indeed a metric on $\mathrm{Cts}(X,X)$ by Theorem L13-2 since X is compact and metrisable, and then $\mathrm{Cts}_\lambda(X,X) \subseteq \mathrm{Cts}(X,X)$, given the subspace topology, also inherits this metric.)

So, fix some $f \in \mathrm{Cts}_\lambda(X,X)$. Note that f is a λ-contraction so for all $x,y \in X$ we have

$$d(f(x), f(y)) \le \lambda\, d(x,y). \tag{1}$$

Now, let $\epsilon > 0$ be given. Let $\delta = \epsilon(1-\lambda)$, which is positive since $\epsilon > 0$ and $\lambda < 1$. Suppose that some $g \in \mathrm{Cts}_\lambda(X,X)$ satisfies $d_\infty(f,g) < \delta$. We want to show that $d(\mathrm{fix}(f), \mathrm{fix}(g)) < \epsilon$. To do this, first let $a = \mathrm{fix}(f)$ and $b = \mathrm{fix}(g)$. Then, by definition of the fix function we have

$$f(a) = a \qquad \text{and} \qquad g(b) = b. \tag{2}$$

And now we have

$$\begin{aligned}
d(a,b) &\le d(a, f(b)) + d(f(b), b) && \text{(by the triangle inequality)} \\
&= d(f(a), f(b)) + d(f(b), g(b)) && \text{(by (2))} \\
&\le \lambda\, d(a,b) + d(f(b), g(b)) && \text{(by (1))} \\
&< \lambda\, d(a,b) + \delta && \text{(since } d(f(b), g(b)) \le \sup_{x\in X}\{d(f(x), g(x))\} = d_\infty(f,g) < \delta\text{)}
\end{aligned}$$

i.e. $d(a,b) < \lambda\, d(a,b) + \delta$. Rearranging, we get $(1-\lambda)d(a,b) < \delta$, which gives (remembering that $1-\lambda > 0$):

$$d(\mathrm{fix}(f), \mathrm{fix}(g)) = d(a,b) < \frac{\delta}{1-\lambda} = \frac{\epsilon(1-\lambda)}{1-\lambda} = \epsilon.$$

So indeed, whenever some $g \in \mathrm{Cts}_\lambda(X,X)$ satisfies $d_\infty(f,g) < \delta$, we have $d(\mathrm{fix}(f), \mathrm{fix}(g)) < \epsilon$.

As we discussed earlier, this shows that the function fix sending a contraction mapping to its unique fixed point is continuous, which is what we wanted.

Exercise L16-6

Let A be the set of polynomials in $\mathrm{Poly}(X,\mathbb{R})$ whose coefficients are all rational (or rather, the set of polynomial functions in $\mathrm{Poly}(X,\mathbb{R})$ which are induced by polynomials with rational coefficients). We claim that A is a countable dense subset of $\mathrm{Cts}(X,\mathbb{R})$. Our proof is in a couple of parts:

A is a countable set: Firstly, to distinguish between polynomials and polynomial functions we'll denote by A' the set of polynomials with rational coefficients. Clearly if A' is countable then A is countable, so we'll just show A' is countable. For each polynomial $p(x) \in A'$, we can uniquely express $p(x)$ in the form (writing $\vec{N} = (N_1, N_2, \ldots, N_n)$)

$$p(x) = \sum_{\vec{N}\in Y} \frac{a_{\vec{N}}}{b_{\vec{N}}} \pi_1(x)^{N_1} \pi_2(x)^{N_2} \ldots \pi_n(x)^{N_n}$$

where $Y \subset \mathbb{N}^n$ is finite, and for each $\vec{N} \in Y$, $a_{\vec{N}}, b_{\vec{N}}$ are nonzero integers such that $b_{\vec{N}} > 0$ and $\gcd(a_{\vec{N}}, b_{\vec{N}}) = 1$ for $\vec{N} \in Y$, and $a_n \neq 0$. For $p(x)$ as above, denote by $N(p(x))$ the quantity

$$N(p(x)) := \sum_{\vec{N} \in Y} |a_{\vec{N}}| + |b_{\vec{N}}| + N_1 + N_2 + \cdots + N_n$$

Then, clearly the set

$$A_k := \{p(x) \in A' \mid N(p(x)) \leq k\}$$

is finite for each integer $k \geq 0$, and clearly $A' = \bigcup_{k=0}^{\infty} A_k$. So we can list out all the polynomials in A by listing out the polynomials in A_0, then A_1, then A_2, and so on. This gives a bijection from $\mathbb{N}$ to A', hence A' is countable. Hence A is also countable.

$A \subseteq \mathbf{Cts}(X, \mathbb{R})$ is dense, i.e. $\overline{A} = \mathbf{Cts}(X, \mathbb{R})$: We'll first show that $\mathrm{Poly}(X, \mathbb{R}) \subseteq \overline{A}$. Let $p \in \mathrm{Poly}(X, \mathbb{R})$, we'll show that p can be uniformly approximated by elements of A. We can write p as

$$p(x) = \sum_{\vec{N} \in Y} c_{\vec{N}} \pi_1(x)^{N_1} \pi_2(x)^{N_2} \ldots \pi_n(x)^{N_n}$$

where $Y \subset \mathbb{N}^n$ is finite, and $c_{\vec{N}} \in \mathbb{R}$ for $\vec{N} \in Y$. Now, since $X \subseteq \mathbb{R}$ is compact, the image of X under the map $\pi_1^{N_1} \pi_2^{N_2} \ldots \pi_n^{N_n}$ (which is continuous since it a product of projections) must be compact. Since $\pi_1^{N_1} \pi_2^{N_2} \ldots \pi_n^{N_n}(X) \subseteq \mathbb{R}$ and compact sets in $\mathbb{R}$ are bounded, it follows that $\pi_1^{N_1} \pi_2^{N_2} \ldots \pi_n^{N_n}(X) \subseteq \mathbb{R}$ must be bounded for each $\vec{N} \in Y$. Since Y is finite, we then get that there must exist some K such that

$$|\pi_1(x)^{N_1} \pi_2(x)^{N_2} \ldots \pi_n(x)^{N_n}| < K \qquad \forall x \in X \text{ and } \vec{N} \in Y \tag{1}$$

Now, since $\overline{\mathbb{Q}} = \mathbb{R}$, for each $\vec{N} \in Y$ we can find some $q_{\vec{N}} \in \mathbb{Q}$ such that

$$|c_{\vec{N}} - q_{\vec{N}}| < \frac{\epsilon}{2|Y|K} \tag{2}$$

(here we are using Exercise L13-4, which tells us that since $c_{\vec{N}} \in \mathbb{R} = \overline{\mathbb{Q}}$, every open neighbourhood of c_i contains an element of $\mathbb{Q}$.) Now consider the polynomial $q \in A$ given by

$$p(x) = \sum_{\vec{N} \in Y} q_{\vec{N}} \pi_1(x)^{N_1} \pi_2(x)^{N_2} \ldots \pi_n(x)^{N_n}.$$

We have, for each $x \in X$,

$$\begin{aligned}
|p(x) - q(x)| &= \left| \sum_{\vec{N} \in Y} c_{\vec{N}} \pi_1(x)^{N_1} \pi_2(x)^{N_2} \ldots \pi_n(x)^{N_n} - \sum_{\vec{N} \in Y} q_{\vec{N}} \pi_1(x)^{N_1} \pi_2(x)^{N_2} \ldots \pi_n(x)^{N_n} \right| \\
&= \left| \sum_{\vec{N} \in Y} (c_{\vec{N}} - q_{\vec{N}}) \pi_1(x)^{N_1} \pi_2(x)^{N_2} \ldots \pi_n(x)^{N_n} \right| \\
&\leq \sum_{\vec{N} \in Y} \left| (c_{\vec{N}} - q_{\vec{N}}) \pi_1(x)^{N_1} \pi_2(x)^{N_2} \ldots \pi_n(x)^{N_n} \right| && \text{(triangle inequality)} \\
&= \sum_{\vec{N} \in Y} \left| (c_{\vec{N}} - q_{\vec{N}}) \right| \cdot \left| \pi_1(x)^{N_1} \pi_2(x)^{N_2} \ldots \pi_n(x)^{N_n} \right| \\
&\leq \sum_{\vec{N} \in Y} \frac{\epsilon}{2|Y|K} \cdot K && \text{(by (1) and (2))} \\
&= \frac{\epsilon}{2} < \epsilon.
\end{aligned}$$

So we have shown that for any $p \in \mathrm{Poly}(X,\mathbb{R})$ and any $\epsilon > 0$ there exists $q \in A$ such that $|p(x) - q(x)| < \epsilon$ for all $x \in X$. By Exercise L16-1 (which we can apply since X is compact, $\mathbb{R}$ is a metric space and $A \subseteq \mathrm{Poly}(X,\mathbb{R}) \subseteq \mathrm{Cts}(X,\mathbb{R})$ we conclude that $p \in \overline{A}$ for every $p \in \mathrm{Poly}(X,\mathbb{R})$, that is,

$$\begin{aligned} &\mathrm{Poly}(X,\mathbb{R}) \subseteq \overline{A} \\ \implies &\overline{\mathrm{Poly}(X,\mathbb{R})} \subseteq \overline{A} \end{aligned}$$

(using the definition of $\overline{\mathrm{Poly}(X,\mathbb{R})}$ and the fact that $\overline{A}$ is closed). But also, Exercise L13-4 tells us that since $A \subseteq \mathrm{Poly}(X,\mathbb{R})$ we also have

$$\overline{A} \subseteq \overline{\mathrm{Poly}(X,\mathbb{R})}$$

so together we see that we must have $\overline{A} = \overline{\mathrm{Poly}(X,\mathbb{R})}$. It remains to note that from Corollary L16-4, which we can use since $X \subseteq \mathbb{R}^n$ is compact, we also have $\overline{\mathrm{Poly}(X,\mathbb{R})} = \mathrm{Cts}(X,\mathbb{R})$, and this gives

$$\overline{A} = \overline{\mathrm{Poly}(X,\mathbb{R})} = \mathrm{Cts}(X,\mathbb{R})$$

and therefore A is a dense subset of $\mathrm{Cts}(X,\mathbb{R})$.

Conclusion (and second countability): We have shown A is a countable dense subset of $\mathrm{Cts}(X,\mathbb{R})$, hence $\mathrm{Cts}(X,\mathbb{R})$ is separable. We'll now show that $\mathrm{Cts}(X,\mathbb{R})$ is second-countable.

First note that $\mathrm{Cts}(X,\mathbb{R})$ is metrisable with the d_∞ metric (by Theorem L13-2, noting that X is compact and $\mathbb{R}$ is metrisable). Consider the set $\mathcal{B} = \{B_\epsilon(p) \mid p \in A, \epsilon \in \mathbb{Q}_{>0}\}$, we claim $\mathcal{B}$ is countable and is also a basis for the topology on $\mathrm{Cts}(X,\mathbb{R})$.

Indeed, $\mathcal{B}$ is countable since we proved previously that A is countable and we also know that $\mathbb{Q}_{>0}$ is countable, which implies that $A \times \mathbb{Q}_{>0}$ is countable (since the Cartesian product of two countable sets is countable- it is easy to see how the usual proof that $\mathbb{Q}$ is countable extends to this), and it follows that $\mathcal{B}$ is also countable.

As for proving $\mathcal{B}$ is a basis for the topology on $\mathrm{Cts}(X,\mathbb{R})$, let $U \subseteq \mathrm{Cts}(X,\mathbb{R})$ be open and $f \in U$. We need to show that there exists $B \in \mathcal{B}$ with $f \in B \subseteq U$. From the definition of the topology on a metric space, since U is open and $f \in U$ we know there exists $\epsilon_1 > 0$ such that $B_{\epsilon_1}(f) \subseteq U$. Now, since $f \in \mathrm{Cts}(X,\mathbb{R}) = \overline{A}$, we know (by Exercise L16-1) that there exists some $q \in A$ such that

$$|f(x) - q(x)| < \frac{\epsilon_1}{3} \quad \forall x \in X \qquad \text{i.e.} \qquad d_\infty(f,q) < \frac{\epsilon_1}{3} \tag{3}$$

Also, since $\frac{\epsilon_1}{2} \in \mathbb{R} = \overline{\mathbb{Q}}$, we know (by Exercise L13-4) that every open neighbourhood of $\frac{\epsilon_1}{2}$ must contain an element of $\mathbb{Q}$, and this implies that there exists some rational number ϵ_2 such that

$$\left|\epsilon_2 - \frac{\epsilon_1}{2}\right| < \frac{\epsilon_1}{6} \qquad \text{i.e.} \qquad -\frac{\epsilon_1}{6} < \epsilon_2 - \frac{\epsilon_1}{2} < \frac{\epsilon_1}{6} \qquad \text{i.e.} \qquad \frac{\epsilon_1}{3} < \epsilon_2 < \frac{2\epsilon_1}{3}.$$

Now we claim that $f \in B_{\epsilon_2}(q) \subseteq U$. Indeed, by (3) we have

$$d_\infty(f,q) < \frac{\epsilon_1}{3} < \epsilon_2$$

which implies that $f \in B_{\epsilon_2}(q)$. And, for any $g \in B_{\epsilon_2}(q)$, we have $d_\infty(q,g) < \epsilon_2 < 2\epsilon_1/3$ which implies

$$\begin{aligned} d_\infty(f,g) &\leq d_\infty(f,q) + d_\infty(q,g) && \text{(triangle inequality)} \\ &< \frac{\epsilon_1}{3} + \frac{2\epsilon_1}{3} && \text{(by (3) and our earlier calculation for } d_\infty(q,g)) \\ &= \epsilon_1 \\ \implies &g \in B_{\epsilon_1}(f) \end{aligned}$$

and since this holds for all $g \in B_{\epsilon_2}(q)$, we have $B_{\epsilon_2}(q) \subseteq B_{\epsilon_1}(f)$, and recalling that $B_{\epsilon_1}(f) \subseteq U$, we get $B_{\epsilon_2}(q) \subseteq U$.

Since $\epsilon_2 \in \mathbb{Q}$ and $q \in A$, taking $B = B_{\epsilon_2}(q)$ we have $B = B_{\epsilon_2}(q) \in \mathcal{B}$ and this set satisfies $f \in B \subseteq U$. We can find such a $B \in \mathcal{B}$ for any open $U \subseteq \mathrm{Cts}(X,\mathbb{R})$ and $f \in U$, so this implies that $\mathcal{B}$ is a basis for $\mathrm{Cts}(X,\mathbb{R})$.

Putting everything together, we have that $\mathcal{B}$ is countable and is a basis for the topology on $\mathrm{Cts}(X,\mathbb{R})$, hence $\mathrm{Cts}(X,\mathbb{R})$ is second-countable.

Exercise L17-7

We want to prove that C (as defined in the question) is finite-dimensional if and only if X is a finite set of points with the discrete topology. To do this, consider the following cases:

Case 1: X is finite: In this case, let the elements of X be $X = \{x_1, x_2, \ldots, x_n\}$. We'll first prove the following lemma:

Lemma 1. *If $X = \{x_1, x_2, \ldots, x_n\}$ is a finite Hausdorff topological space, then X has the discrete topology.*

Proof. First, fix some i $(1 \leq i \leq n)$, we will show that $\{x_i\}$ is open. Since X is Hausdorff, for each j such that $j \neq i$ we know that there must exist open sets U_j and V_j such that

$$x_i \in U_j, \qquad x_j \in V_j, \qquad U_j \cap V_j = \emptyset.$$

Note that $x_j \in V_j$ and $U_j \cap V_j = \emptyset$ implies that $x_j \notin U_j$. Now consider the finite intersection

$$U = \bigcap_{j \neq i} U_j.$$

U must be open it is a finite intersection of open sets. Moreover, since $x_i \in U_j$ for all $j \neq i$, we must have $x_i \in U$. And, since $x_j \notin U_j$ for each $j \neq i$, we also have $x_j \notin U$. So U is a subset of X containing x_i but not containing x_j for $j \neq i$, hence $U = \{x_i\}$. Then, since U is open, we have that $\{x_i\}$ must also be open, and this is true for every $1 \leq i \leq n$.

Now, any other subset $S \subseteq X$ can be written in the form

$$S = \bigcup_{i \in I} \{x_i\}$$

where $I \subseteq \{1, 2, \ldots, n\}$, and then since a union of open sets is open, it follows that S must be open. Hence all subsets of X are open and X must have the discrete topology. □

Applying the lemma to this case (which we can do since X is finite and Hausdorff), we get that X has the discrete topology. This implies that every function $g \colon X \to \mathbb{R}$ is continuous, hence $\mathrm{Cts}(X, \mathbb{R})$ is just the set of all functions from X to $\mathbb{R}$.

Now, for $1 \leq i \leq n$ define $f_i \colon X \to \mathbb{R}$ as follows:

$$f_i(x_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

We claim that $\{f_1, f_2, \ldots, f_n\}$ span C. Indeed, for any $g \in C$, we have

$$\begin{aligned} g(x_j) &= \sum_{i=0}^{n} g(x_i) f_i(x_j) \qquad \forall\, j \in \{1, 2, \ldots, n\} \\ \implies g &= \sum_{i=0}^{n} g(x_i) f_i \end{aligned}$$

(where $g(x_i) f_i$ is the function f_i multiplied by the scalar $g(x_i)$). So any $g \in C$ can be expressed as a linear combination of f_is, hence C is spanned by $\{f_1, f_2, \ldots, f_n\}$ as claimed.

Since C is spanned by a finite set, C must be finite-dimensional. This proves that if X is a finite set of points with the discrete topology, then C is finite-dimensional.

Case 2: X is infinite: We will show that if X is infinite, then C must be infinite-dimensional. Suppose otherwise, for the sake of contradiction, that C is finite-dimensional. Let $n = \dim C$. Since X is infinite, we can find $n+1$ distinct points in X, say, $x_1, x_2, \ldots, x_{n+1} \in X$. Now, we will prove two lemmas about X:

Lemma 2. *There exist pairwise disjoint open sets U_i, $1 \leq i \leq n+1$, such that $x_i \in U_i$ for $1 \leq i \leq n+1$.*

Proof. Since X is Hausdorff, for any $1 \leq i < j \leq n$ there exist open sets $V_{i,j}, W_{i,j} \subseteq X$ such that

$$x_i \in V_{i,j}, \qquad x_j \in W_{i,j}, \qquad V_{i,j} \cap W_{i,j} = \emptyset.$$

Now, for each $1 \leq i \leq n+1$, define U_i as follows:

$$U_i = \left(\bigcap_{j<i} W_{j,i} \right) \cap \left(\bigcap_{j>i} V_{i,j} \right).$$

For all $j < i$ we have $x_i \in W_{j,i}$, and for all $j > i$ we have $x_i \in V_{i,j}$, therefore $x_i \in U_i$. And, for $i < j$ we have

$$\begin{aligned} U_i \cap U_j &= \left(\bigcap_{k<i} W_{k,i} \right) \cap \left(\bigcap_{k>i} V_{i,k} \right) \cap \left(\bigcap_{k<j} W_{k,j} \right) \cap \left(\bigcap_{k>j} V_{j,k} \right) \\ &\subseteq V_{i,j} \cap W_{i,j} = \emptyset \end{aligned}$$

and this implies that $U_i \cap U_j = U_j \cap U_i = \emptyset$. So indeed, the U_i's are pairwise disjoint and satisfy $x_i \in U_i$ for $1 \leq i \leq n+1$. ☐

Lemma 3. *For each $x \in X$, the set $\{x\}$ is closed.*

Proof. Fix any $x \in X$. Since X is Hausdorff, for any $y \in X \backslash \{x\}$ there exist open sets $U_y, V_y \subseteq X$ such that

$$x \in U_y, \qquad y \in V_y, \qquad U_y \cap V_y = \emptyset.$$

Note that since $x \in U_y$ and $U_y \cap V_y = \emptyset$, we must have $x \notin V_y$. So now, consider

$$V = \bigcup_{y \in X \backslash \{x\}} V_y.$$

V is open since it is a union of open sets (each V_y was open). Also, since $x \notin V_y$ for all $y \in X \backslash \{x\}$, we have $x \notin V$. And, since $y \in V_y$ for all $y \in X \backslash \{x\}$, we have $y \in V$ for all $y \in X \backslash \{x\}$. Since $x \notin V$ and $y \in V$ for all $y \in X \backslash \{x\}$, we must have $V = X \backslash \{x\}$, then since V is open $X \backslash \{x\}$ must also be open. This implies $\{x\}$ is closed, as we wanted to show. ☐

From the above two lemmas, we know that $\{x_i\}$ is closed for each $1 \leq i \leq n+1$, and we also know that there exist pairwise disjoint open sets U_i, $1 \leq i \leq n+1$ such that $x_i \in U_i$ for $1 \leq i \leq n+1$.

Now, for each $1 \leq i \leq n+1$ we will define a function $f_i \colon X \to \mathbb{R}$ as follows. Let $A_i = \{x_i\}$ and $B_i = X \backslash U_i$. As mentioned before, $A_i = \{x_i\}$ is closed, and B_i is also closed since U_i is open. Moreover, since $x_i \in U_i$, we have $x_i \notin X \backslash U_i = B_i$ so that A_i and B_i are disjoint. Also, since X is compact Hausdorff, it is normal (by Exercise L11-9). So, we may use the Urysohn lemma to get that there exists a continuous function $f_i \colon X \to \mathbb{R}$ such that

$$f_i(a) = 1 \quad \forall\, a \in A_i \qquad \text{and} \qquad f_i(b) = 0 \quad \forall\, b \in B_i$$

Since $A_i = \{x_i\}$, this implies that $f_i(x_i) = 1$. Also, since $U_i \cap U_j = \emptyset$ and $x_j \in U_j$ for $j \neq i$, we have $x_j \notin U_i$, which implies $x_j \in X \backslash U_i = B_i$, for $j \neq i$. So, for each $1 \leq i \leq n+1$ we have

$$f_i(x_i) = 1 \qquad \text{and} \qquad f_i(x_j) = 0 \quad \text{if } j \neq i \tag{1}$$

Now we'll show that these f_is are linearly independent. Indeed, suppose that there exist $\alpha_1, \alpha_2, \ldots, \alpha_{n+1}$ such that

$$\begin{aligned}
\sum_{i=1}^{n+1} \alpha_i f_i &= 0 \\
\implies \sum_{i=1}^{n+1} \alpha_i f_i(x_j) &= 0 \quad \forall j \in \{1, 2, \ldots, n+1\} \\
\implies \alpha_j &= 0 \quad \forall j \in \{1, 2, \ldots, n+1\} \qquad \text{(by (1))}
\end{aligned}$$

This shows that the set $\{f_1, f_2, \ldots, f_{n+1}\}$ is linearly independent. But now we have a contradiction, as C, which is of dimension n, cannot contain a set of $n+1$ linearly independent vectors. Hence our initial supposition was wrong, and C must be infinite dimensional.

Conclusion: From our cases we see that the only situation where C is finite-dimensional is when X is finite with the discrete topology, hence C is finite-dimensional if and only if X is a finite set of points with the discrete topology.

Exercise L18-7

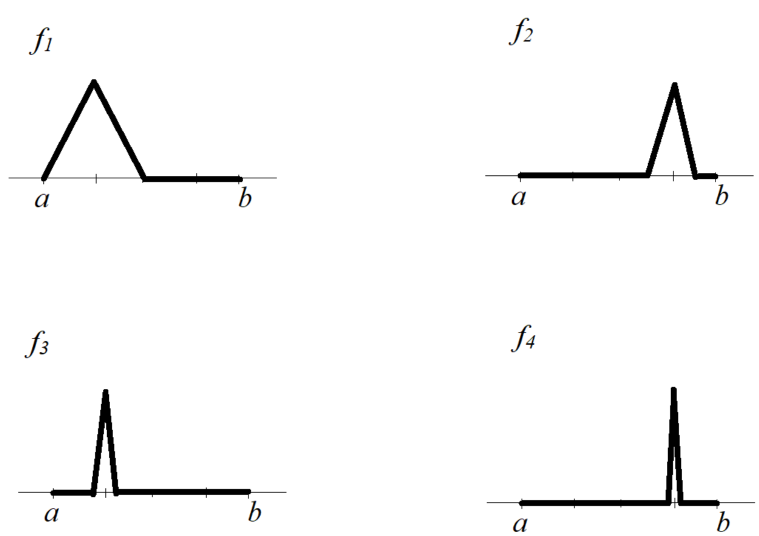
Let $a, b \in \mathbb{R}$ with $a < b$ (short comment about $a = b$ case is at end of this solution). Consider the sequence of functions $f_n \colon [a, b] \to \mathbb{R}$ given for $n \geq 1$ by

$$f_n(x) = \begin{cases} 0 & \text{if } x \in [a, c_n - d_n] \cup [c_n + d_n, b] \\ \frac{1}{d_n}(x - c_n + d_n) & \text{if } x \in (c_n - d_n, c_n) \\ -\frac{1}{d_n}(x - c_n - d_n) & \text{if } x \in [c_n, c_n + d_n) \end{cases}$$

where

$$c_n = \begin{cases} a + \frac{b-a}{4} & \text{if } n \text{ odd} \\ a + \frac{3(b-a)}{4} & \text{if } n \text{ even} \end{cases} \qquad \text{and} \qquad d_n = \frac{b-a}{2^{n+1}}$$

To illustrate, here are some sketches of the first few functions in this sequence:



We'll now show that $f_n \to f$, where f is given by $f(x) = 0$ for all $x \in [a,b]$. Indeed, for $n \geq 1$ note that f_n and f are Riemann integrable since it is continuous so we have

$$\begin{aligned}
d_p(f_n, f) = \|f_n - f\|_p &= \left\{ \int_{[a,b]} |f_n - f|^p \right\}^{1/p} = \left\{ \int_a^b |f_n(x)|^p \, dx \right\}^{1/p} \\
&= \left\{ \int_a^{c_n - d_n} |f_n(x)|^p \, dx + \int_{c_n - d_n}^{c_n + d_n} |f_n(x)|^p \, dx + \int_{c_n + d_n}^{b} |f_n(x)|^p \, dx \right\}^{1/p} = \left\{ 0 + \int_{c_n - d_n}^{c_n + d_n} |f_n(x)|^p \, dx + 0 \right\}^{1/p} \\
&\leq \left\{ \int_{c_n - d_n}^{c_n + d_n} |1|^p \, dx \right\}^{1/p} = \left\{ \int_{c_n - d_n}^{c_n + d_n} 1 \, dx \right\}^{1/p} = (2d_n)^{1/p} = \left(\frac{b-a}{2^n} \right)^{1/p}
\end{aligned}$$

which goes to zero as $n \to \infty$, prove that $f_n \to f$ as $n \to \infty$. (Or, if I need to give an ϵ-N argument: for any $\epsilon > 0$ take $N > \log_2(\frac{b-a}{\epsilon^p})$, then for $n \geq N$ the above shows that $d_p(f_n, f) \leq \left(\frac{b-a}{2^n}\right)^{1/p} \leq \left(\frac{b-a}{2^N}\right)^{1/p} < \epsilon$).

So we have proved that f_n converges in $(\mathrm{Cts}([a,b], \mathbb{R}), d_p)$. However, f_n does not converge pointwise, since if we consider the point $z = a + \frac{b-a}{4}$, we have $f_n(z) = 1$ if n is odd and $f_n(z) = 0$ if n is even, so the sequence $(f_n(z))_{n=0}^{\infty}$ does not converge.

Thus convergence $f_n \to f$ in $(\mathrm{Cts}([a,b], \mathbb{R}), d_p)$ for $1 \leq p < \infty$ does not imply pointwise convergence.

(A comment about the $a = b$ case if it's needed: This case is trivial to deal with since every sequence of functions in $(\mathrm{Cts}([a,a], \mathbb{R}), d_p)$ will converge to the zero function (the associated integral is always zero), but for example the sequence of functions $f_n \colon [a,b] \to \mathbb{R}$ given by $f_n(a) = n$ for $n \geq 1$ is not pointwise convergent. So convergence still doesn't imply pointwise convergence.)

Exercise L18-12

Denote by $\Psi \colon (\widehat{X \times Y}) \to \hat{X} \times \hat{Y}$ the function sending

$$((x_n, y_n))_{n=0}^{\infty} \mapsto ((x_n)_{n=0}^{\infty}, (y_n)_{n=0}^{\infty}).$$

We want to prove that Ψ is a well-defined distance preserving bijection. Denote by $\hat{d}$ the metric on $(\widehat{X \times Y})$, by $\hat{d_X}$ the metric on $\hat{X}$, by $\hat{d_Y}$ the metric on $\hat{Y}$, by d the metric on $X \times Y$, and by d' the metric on $\hat{X} \times \hat{Y}$.

Ψ is well-defined: Let $((x_n, y_n))_{n=0}^{\infty} \in (\widehat{X \times Y})$. We'll first show that $((x_n)_{n=0}^{\infty}, (y_n)_{n=0}^{\infty}) \in \hat{X} \times \hat{Y}$, i.e. that $(x_n)_{n=0}^{\infty}$ and $(y_n)_{n=0}^{\infty}$ are Cauchy sequences. Let $\epsilon > 0$ be given. Since $((x_n, y_n))_{n=0}^{\infty}$ is Cauchy, there exists N such that for all $n, m \geq N$ we have

$$\begin{aligned}
& d((x_n, y_n), (x_m, y_m)) < \epsilon \\
\implies & d_X(x_n, x_m) + d_Y(y_n, y_m) < \epsilon && \text{(by definition of product metric of } X \times Y \\
\implies & d_X(x_n, x_m) < \epsilon \text{ and } d_Y(y_n, y_m) < \epsilon && \text{(since } d_X(x_n, x_m), d_Y(y_n, y_m) \geq 0)
\end{aligned}$$

and it follows that $(x_n)_{n=0}^{\infty}$ and $(y_n)_{n=0}^{\infty}$ are Cauchy sequences.

The other thing we have to check is that if $((x_n, y_n))_{n=0}^{\infty}, (a_n, b_n))_{n=0}^{\infty} \in (\widehat{X \times Y})$ such that $((x_n, y_n))_{n=0}^{\infty} \sim ((a_n, b_n))_{n=0}^{\infty}$, then $((x_n)_{n=0}^{\infty}, (y_n)_{n=0}^{\infty}) = ((a_n)_{n=0}^{\infty}, (b_n)_{n=0}^{\infty})$, i.e. $(x_n)_{n=0}^{\infty} \sim (a_n)_{n=0}^{\infty}$ and $(y_n)_{n=0}^{\infty} \sim (b_n)_{n=0}^{\infty}$. We have

$$\begin{aligned}
& ((x_n, y_n))_{n=0}^{\infty} \sim ((a_n, b_n))_{n=0}^{\infty} \\
\implies & \lim_{n \to \infty} d((x_n, y_n), (a_n, b_n)) = 0 && \text{(by definition of } \sim) \\
\implies & \lim_{n \to \infty} d_X(x_n, a_n) + d_Y(y_n, b_n) = 0 && \text{(by definition of product metric of } X \times Y)
\end{aligned}$$

Now, since $0 \le d_X(x_n, a_n) \le d_X(x_n, a_n) + d_Y(y_n, b_n)$ and $0 \le d_Y(y_n, b_n) \le d_X(x_n, a_n) + d_Y(y_n, b_n)$ for each $n \ge 1$, the above (together with Sandwich Theorem) implies

$$\begin{aligned}
\lim_{n\to\infty} d_X(x_n, a_n) = 0 \qquad &\text{and} \qquad \lim_{n\to\infty} d_Y(y_n, b_n) = 0 \\
\implies (x_n)_{n=0}^{\infty} \sim (a_n)_{n=0}^{\infty} \qquad &\text{and} \qquad (y_n)_{n=0}^{\infty} \sim (b_n)_{n=0}^{\infty} \\
\implies ((x_n)_{n=0}^{\infty}, (y_n)_{n=0}^{\infty}) &= ((a_n)_{n=0}^{\infty}, (b_n)_{n=0}^{\infty})
\end{aligned}$$

hence Ψ is well-defined.

Ψ is distance-preserving: For $((x_n, y_n))_{n=0}^{\infty}, ((a_n, b_n))_{n=0}^{\infty} \in (\widehat{X \times Y})$, we have

$$\begin{aligned}
& d'(\Psi(((x_n, y_n))_{n=0}^{\infty}), \Psi(((a_n, b_n))_{n=0}^{\infty})) \\
&= d'(((x_n)_{n=0}^{\infty}, (y_n)_{n=0}^{\infty}), ((a_n)_{n=0}^{\infty}, (b_n)_{n=0}^{\infty})) \\
&= \hat{d_X}((x_n)_{n=0}^{\infty}, (a_n)_{n=0}^{\infty}) + \hat{d_Y}((y_n)_{n=0}^{\infty}, (b_n)_{n=0}^{\infty}) && \text{(by definition of product metric of } \hat{X} \times \hat{Y}) \\
&= \lim_{n\to\infty} d_X(x_n, a_n) + \lim_{n\to\infty} d_Y(y_n, b_n) && \text{(by definition of completion of } X, Y) \\
&= \lim_{n\to\infty} d_X(x_n, a_n) + d_Y(y_n, b_n) \\
&= \lim_{n\to\infty} d((x_n, y_n), (a_n, b_n)) && \text{(by definition of product metric of } X \times Y) \\
&= \hat{d}(((x_n, y_n))_{n=0}^{\infty}, ((a_n, b_n))_{n=0}^{\infty}) && \text{(by definition of completion of } X \times Y
\end{aligned}$$

hence Ψ is distance preserving.

Ψ is injective: Suppose $((x_n, y_n))_{n=0}^{\infty}, ((a_n, b_n))_{n=0}^{\infty} \in (\widehat{X \times Y})$ such that $\Psi(((x_n, y_n))_{n=0}^{\infty}) = \Psi(((a_n, b_n))_{n=0}^{\infty})$. Then

$$\begin{aligned}
& \Psi(((x_n, y_n))_{n=0}^{\infty}) = \Psi(((a_n, b_n))_{n=0}^{\infty}) \\
& \implies ((x_n)_{n=0}^{\infty}, (y_n)_{n=0}^{\infty}) = ((a_n)_{n=0}^{\infty}, (b_n)_{n=0}^{\infty}) \\
& \implies (x_n)_{n=0}^{\infty} \sim (a_n)_{n=0}^{\infty} \qquad \text{and} \qquad (y_n)_{n=0}^{\infty} \sim (b_n)_{n=0}^{\infty} \\
& \implies \lim_{n\to\infty} d_X(x_n, a_n) = 0 \qquad \text{and} \qquad \lim_{n\to\infty} d_Y(y_n, b_n) = 0 \\
& \implies \lim_{n\to\infty} d_X(x_n, a_n) + d_Y(y_n, b_n) = 0 \\
& \implies \lim_{n\to\infty} d((x_n, y_n), (a_n, b_n)) = 0 && \text{(by definition of product metric of } X \times Y) \\
& \implies ((x_n, y_n))_{n=0}^{\infty} \sim ((a_n, b_n))_{n=0}^{\infty}
\end{aligned}$$

which implies that $((x_n, y_n))_{n=0}^{\infty}, ((a_n, b_n))_{n=0}^{\infty}$ are equal as elements (equivalence classes) in $(\widehat{X \times Y})$. Hence Ψ is injective.

Ψ is surjective: Every element in $\hat{X} \times \hat{Y}$ can be written in the form $((x_n)_{n=0}^{\infty}, (y_n)_{n=0}^{\infty})$ where $(x_n)_{n=0}^{\infty} \in \hat{X}$ and $(y_n)_{n=0}^{\infty} \in \hat{Y}$. We just have to show $((x_n, y_n))_{n=0}^{\infty} \in (\widehat{X \times Y})$, as we would then have $\Psi(((x_n, y_n))_{n=0}^{\infty}) = ((x_n)_{n=0}^{\infty}, (y_n)_{n=0}^{\infty})$. Indeed, since $(x_n)_{n=0}^{\infty} \in \hat{X}$ and $(y_n)_{n=0}^{\infty} \in \hat{Y}$, we have that $(x_n)_{n=0}^{\infty}$ and $(y_n)_{n=0}^{\infty}$ are Cauchy sequences, and this implies that for any $\epsilon > 0$ there exist N_1 and N_2 such that $d_X(x_n, x_m) < \epsilon/2$ when $n, m \ge N_1$ and $d_Y(y_n, y_m) < \epsilon/2$ when $n, m \ge N_2$. Taking $N = \max(N_1, N_2)$, for all $n, m \ge N$ we have

$$\begin{aligned}
& d_X(x_n, x_m) < \epsilon/2 \text{ and } d_Y(y_n, y_m) < \epsilon/2 \\
& \implies d_X(x_n, x_m) + d_Y(y_n, y_m) < \epsilon \\
& \implies d((x_n, y_n), (x_m, y_m)) < \epsilon && \text{(by definition of product metric of } X \times Y
\end{aligned}$$

Hence $((x_n, y_n))_{n=0}^{\infty}$ is a Cauchy sequence, so $((x_n, y_n))_{n=0}^{\infty} \in (\widehat{X \times Y})$. As discussed, this implies that Ψ is surjective.

Conclusion: We have shown Ψ is well-defined, distance-preserving, injective and surjective, hence Ψ is a distance preserving bijection.