MAST30026 Metric and Hilbert Spaces

Assignment 2 Solutions — 2018 Semester 2

Total available marks: 22

Notation and conventions

The set $\mathbb{N}$ is the set of positive integers $\{1, 2, 3, \dots\}$.

The set $\mathbb{N}_0$ is the set of nonnegative integers $\{0, 1, 2, \dots\}$.

Question 1 (1 mark)

Exercise L5-7. Determine the hyperbolic angle θ such that the Lorentz boost F from p. 13 is a hyperbolic rotation H_θ. That is, given $0 \leqslant r < c$ and $\gamma = (1-r^2)^{-1/2}$, solve (here we set $c = 1$)

$$\begin{bmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{bmatrix} = \begin{bmatrix} \gamma & \gamma r \\ \gamma r & \gamma \end{bmatrix},$$

for θ. This shows that the geometry that we have extracted from Einstein's postulates is precisely hyberbolic geometry (at least in the (t,x) plane).

$$\begin{aligned}
\cosh\theta = \gamma &= \frac{1}{\sqrt{1-r^2}} \\
e^\theta + e^{-\theta} &= \frac{2}{\sqrt{1-r^2}} \\
e^{2\theta} - \frac{2}{\sqrt{1-r^2}} e^\theta + 1 &= 0 \\
\left(e^\theta - \frac{1}{\sqrt{1-r^2}}\right)^2 + 1 - \frac{1}{1-r^2} &= 0 \\
\left(e^\theta - \frac{1}{\sqrt{1-r^2}}\right)^2 &= \frac{r^2}{1-r^2} \\
e^\theta - \frac{1}{\sqrt{1-r^2}} &\in \left\{\frac{r}{\sqrt{1-r^2}}, \frac{-r}{\sqrt{1-r^2}}\right\} \\
e^\theta &\in \left\{\frac{1+r}{\sqrt{1-r^2}}, \frac{1-r}{\sqrt{1-r^2}}\right\}
\end{aligned}$$

Now

$$\frac{1-r}{\sqrt{1-r^2}} = \frac{1-r}{\sqrt{1-r}\sqrt{1+r}} = \sqrt{\frac{1-r}{1+r}} < 1,$$

while

$$\frac{1+r}{\sqrt{1-r^2}} = \sqrt{\frac{1+r}{1-r}} > 1.$$

Since $\sinh\theta = \gamma r = r(1-r^2)^{-1/2} > 0$, we know that $\theta > 0$ and $e^\theta > 1$. Hence

$$e^\theta = \sqrt{\frac{1+r}{1-r}} \quad \Longleftrightarrow \quad \theta = \frac{1}{2}\log\left(\frac{1+r}{1-r}\right).$$

Thus the only possible solution is given by $\theta = 1/2\log((1+r)/(1-r))$. Check that this is in fact a solution:

$$\begin{aligned}
\cosh\theta &= \frac{e^\theta + e^{-\theta}}{2} = \frac{1}{2}\left(\sqrt{\frac{1+r}{1-r}} + \sqrt{\frac{1-r}{1+r}}\right) \\
&= \frac{2}{2\sqrt{1-r^2}} = \frac{1}{\sqrt{1-r^2}} = \gamma. \\
\sinh\theta &= \frac{e^\theta - e^{-\theta}}{2} = \frac{1}{2}\left(\sqrt{\frac{1+r}{1-r}} - \sqrt{\frac{1-r}{1+r}}\right) \\
&= \frac{2r}{2\sqrt{1-r^2}} = \frac{r}{\sqrt{1-r^2}} = \gamma r.
\end{aligned}$$

Question 2 (3 marks)

Exercise L6-3. Prove that $X = \{0, 1\}$ with $\mathcal{T} = \{\emptyset, X, \{1\}\}$ is a topological space. This is called the *Sierpiński space* and is usually denoted Σ. Prove Σ is *not* metrisable.

Begin by showing $\mathcal{T}$ is a topology on X.

By inspection $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$.

Suppose $V_1, V_2 \in \mathcal{T}$. We wish to check that $V_1 \cap V_2 \in \mathcal{T}$. We may assume V_1 and V_2 are distinct, since if $V_1 = V_2$ we have $V_1 \cap V_2 = V_1 \in \mathcal{T}$. Furthermore, if either of V_1 or V_2 is $\emptyset \in \mathcal{T}$ then $V_1 \cap V_2 = \emptyset \in \mathcal{T}$. Thus assume V_1 and V_2 are distinct and that neither is $\emptyset \in \mathcal{T}$. Then it is only possible to have $V_1 = \{1\}$ and $V_2 = X$ or $V_1 = X$ and $V_2 = \{1\}$. In both of these cases $V_1 \cap V_2 = \{1\} \in \mathcal{T}$.

Let $U \subseteq \mathcal{T}$ be arbitrary. We wish to check that $\bigcup U \in \mathcal{T}$. If U is empty then $\bigcup U = \emptyset \in \mathcal{T}$. If $X \in U$ then $\bigcup U = X \in \mathcal{T}$. Thus assume U is nonempty and $X \notin U$.

$$\begin{aligned} U = \{\emptyset\} \quad &\implies \quad \bigcup U = \emptyset \in \mathcal{T}. \\ U = \{\{1\}\} \quad &\implies \quad \bigcup U = \{1\} \in \mathcal{T}. \\ U = \{\emptyset, \{1\}\} \quad &\implies \quad \bigcup U = \{1\} \in \mathcal{T}. \end{aligned}$$

This shows that $\mathcal{T} = \{\emptyset, X, \{1\}\}$ is a topology on $X = \{0, 1\}$.

To show that the space Σ is not metrisable, we will assume the existence of a metric $d \colon X \times X \longrightarrow \mathbb{R}$ inducing $\mathcal{T}$ as the topology on X and then reach a contradiction. Let $\varepsilon := d(0, 1)$. Since $0 \neq 1$, we know that $\varepsilon > 0$. Consider the open ball

$$B := \{y \in X \mid d(0, y) < \varepsilon/2\}$$

centred at 0. We know that $0 \in B$ and $1 \notin B$. Since the metric d induces the topology $\mathcal{T}$, the set B must be in $\mathcal{T}$. However, no set in $\mathcal{T}$ contains 0 while not containing 1, so $B \notin \mathcal{T}$. We have reached a contradiction, so indeed the space Σ is not metrisable.

Question 3 (2 marks)

Lemma L7-1. Let X be a set and $\mathcal{B}$ a collection of subsets of X satisfying

(B1) For each $x \in X$ there exists $B \in \mathcal{B}$ with $x \in B$.

(B2) Given $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$ there exists $B_3 \in \mathcal{B}$ with $x \in B_3 \subseteq B_1 \cap B_2$.

Then there is a *unique* topology $\mathcal{T}$ on X for which $\mathcal{B}$ is a basis. We call $\mathcal{T}$ the topology *generated by* $\mathcal{B}$.

Definition. Let $\{X_i\}_{i\in I}$ be an indexed family of topological spaces. The *product space* $\prod_{i\in I} X_i$ is the usual product set with the topology generated by the basis consisting of sets

$$\prod_{i\in I} U_i = \{(x_i)_i \mid x_i \in U_i \text{ for all } i\}$$

where each $U_i \subseteq X_i$ is open and $U_i \neq X_i$ for only finitely many $i \in I$.

Exercise L7-2. Prove that the $\prod_i U_i$ as defined above satisfy (B1), (B2), so that the topology on $\prod_{i\in I} X_i$ is well-defined.

If we take $U_i = X_i$ for every $i \in I$, then we see that the set $\prod_{i\in I} U_i = \prod_{i\in I} X_i$ is in the basis. In this case $U_i \neq X_i$ for no $i \in I$ at all, but nevertheless this is still finitely many $i \in I$. This shows that (B1) is satisfied: For each $x \in \prod_{i\in I} X_i$ we may choose the set $\prod_{i\in I} X_i$ in the basis, and we see that $\prod_{i\in I} X_i$ is a set in the basis containing x.

Next we show (B2). For every $i \in I$ let U_i and V_i be open sets in X_i such that $U_i \neq X_i$ for only finitely many $i \in I$ and $V_i \neq X_i$ for only finitely many $i \in I$. Suppose $x = (x_i)_{i\in I} \in (\prod_{i=I} U_i) \cap (\prod_{i\in I} V_i)$. We wish to produce a set in the basis which contains x and is contained in $(\prod_{i=I} U_i) \cap (\prod_{i\in I} V_i)$. Observe that

$$\left(\prod_{i=I} U_i\right) \cap \left(\prod_{i\in I} V_i\right) = \prod_{i=I} (U_i \cap V_i).$$

We claim that $\prod_{i=I}(U_i \cap V_i)$ is a set in the basis. We can see that $U_i \cap V_i \subseteq X_i$ is open for each $i \in I$ (since U_i and V_i are both open), so it suffices to show that $U_i \cap V_i \neq X_i$ for finitely many $i \in I$.

Now $U_i \cap V_i \neq X_i$ if and only if at least one of U_i and V_i is not X_i. Alternatively, $U_i \cap V_i = X_i$ if and only if $U_i = V_i = X_i$. Let $S := \{i \in I \mid U_i \neq X_i\}$ and $T := \{i \in I \mid V_i \neq X_i\}$. We know that S and T are finite subsets of I. From the discussion above, we know that

$$\{i \in I \mid U_i \cap V_i \neq X_i\} = S \cup T,$$

and $S \cup T$ is a finite subset of I since each of S and T is a finite subset of I. Hence $U_i \cap V_i \neq X_i$ for finitely many $i \in I$.

This shows that

$$\left(\prod_{i=I} U_i\right) \cap \left(\prod_{i\in I} V_i\right) = \prod_{i=I} (U_i \cap V_i)$$

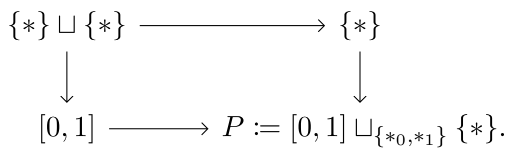
is a set in the basis. Now observe that we have

$$x \in \prod_{i=I} (U_i \cap V_i) \subseteq \left(\prod_{i=I} U_i\right) \cap \left(\prod_{i\in I} V_i\right).$$

We have thus produced a set in the basis which contains x and is contained in $(\prod_{i=I} U_i) \cap (\prod_{i\in I} V_i)$. (Of course, we actually know that the set in the basis is precisely equal to $(\prod_{i=I} U_i) \cap (\prod_{i\in I} V_i)$.) Hence (B2) is satisfied also.

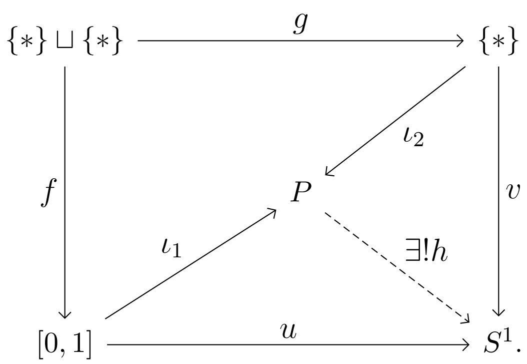
Question 4 (4 marks)

Exercise L7-10. Let us write $\{*_0, *_1\} := \{*\} \sqcup \{*\}$ and $f\colon \{*_0, *_1\} \longrightarrow [0,1]$ for the inclusion of the endpoints $f(*_0) = 0$, $f(*_1) = 1$. We may form the pushout



Prove that $P \cong S^1$.

Consider the following diagram. We will prove that a continuous map $h\colon P \longrightarrow S^1$ exists such that the diagram commutes and then show that h is bijective and open. This will be sufficient to show that $P \cong S^1$.



Here, f, g, ι_1, and ι_2 are the maps in the original commutative diagram, while $u\colon [0,1] \longrightarrow S^1$ and $v\colon \{*\} \longrightarrow S^1$ are given by

$$u(t) := (\cos(2\pi t), \sin(2\pi t)), \quad \forall t \in [0,1]$$
$$v(*) := (1,0).$$

Let $T := (1,0) \in S^1$, and let $X := [0,1] \sqcup \{*\}$. Note that P is a quotient space of X.

Such a map $h\colon P \longrightarrow S^1$ exists and is continuous

We will use the universal property of the pushout to induce h.

We claim that u and v are continuous. That v is continuous can be seen because the preimage of any subset of S^1 (but open subsets in particular) under v is either $\{*\}$ or $\emptyset$, both of which are open.

We now argue that u is continuous: Recall that a basis for the topology on S^1 is the set of arcs along the circle connecting two distinct points on the circle but excluding those two endpoints. (We will call such arcs *open arcs*.) We need only check that the preimage under u of an open arc is an open subset of $[0,1]$.

Let ω_1 and ω_2 be distinct points of S^1. Let θ_1 and θ_2 be the directed and counterclockwise angles subtended at the origin from T to ω_1 and ω_2 respectively. That is,

$$\omega_1 = (\cos\theta_1, \sin\theta_1), \qquad \omega_2 = (\cos\theta_2, \sin\theta_2), \quad \text{and} \quad \theta_1, \theta_2 \in [0, 2\pi).$$

Without loss of generality, assume that $\theta_1 < \theta_2$. There are two open arcs with endpoints ω_1 and ω_2: one beginning at ω_1 and traversed counterclockwise to ω_2 and another beginning at ω_2 and traversed counterclockwise to ω_1. The first (counterclockwise from ω_1) is

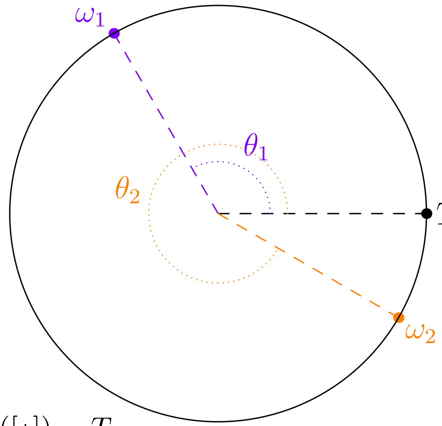
$$\{(\cos\theta, \sin\theta) \mid \theta_1 < \theta < \theta_2\} \subseteq S^1,$$

4. the preimage of which under u is $(\theta_1/(2\pi), \theta_2/(2\pi))$, which is an open subset of $(0,1)$. The second (counterclockwise from ω_2) is

$$\{(\cos\theta, \sin\theta) \mid \theta \in [0,\theta_1) \cup (\theta_2, 2\pi)\} \subseteq S^1,$$

the preimage of which under u is $[0, \theta_1/(2\pi)) \cup (\theta_2/(2\pi), 1]$, which is an open subset of $(0,1)$. Since ω_1 and ω_2 were arbitrary distinct points of S^1, we have shown that u is continuous.

Next, observe that $u \circ f = v \circ g$ as maps from $\{*\} \sqcup \{*\}$ to S^1.



$$\begin{aligned} u(f(*_0)) &= u(0) = T \qquad & u(f(*_1)) &= u(1) = T \\ v(g(*_0)) &= v(*) = T & v(g(*_1)) &= v(*) = T \\ &= u(f(*_0)). & &= u(f(*_1)). \end{aligned}$$

Since $u \circ f = v \circ g$, by the universal property of the pushout there exists a continuous map $h\colon P \longrightarrow S^1$ such that

$$u = h \circ \iota_1 \quad \text{and} \quad v = h \circ \iota_2.$$

These equalities in fact specify h completely: It must be that

$$h([t]) = (\cos(2\pi t), \sin(2\pi t)) \quad \forall 0 < t < 1 \quad \text{and} \quad h([0]) = h([1]) = h([*]) = T.$$

(We use $[\,\cdot\,]$ to denote the equivalence class in P of an element from $X = [0,1] \sqcup \{*\}$.) Let us note that the equivalence classes comprising P induce the following partition of X:

$$\{\{0,1\} \sqcup \{*\}\} \cup \{\{t\} \mid 0 < t < 1\}.$$

$h\colon P \longrightarrow S^1$ is bijective

Observe that

$$h([s]) \neq h([t]) \quad \text{if } 0 < s < t < 1 \quad \text{and} \quad h([t]) \neq T = h([0]) \quad \text{if } 0 < t < 1$$

so we can see that h is injective. Furthermore, h is surjective since

$$S^1 = \{(\cos(2\pi t), \sin(2\pi t)) \mid 0 \leqslant t < 1\} \quad \text{and} \quad (\cos(2\pi t), \sin(2\pi t)) = h([t]) \quad \forall 0 \leqslant t < 1.$$

Thus h is a bijection.

$h\colon P \longrightarrow S^1$ is open

Since $h\colon P \longrightarrow S^1$ is a continuous bijection, in order to show that $P \cong S^1$, it is sufficient to show that h is an open map. Let $q\colon X \longrightarrow P$ be the quotient map.

Take an arbitrary open set $U \subseteq P$, and take an arbitrary point $\omega \in h(U) \subseteq S^1$. We will show that $h(U)$ contains an open neighbourhood of ω.

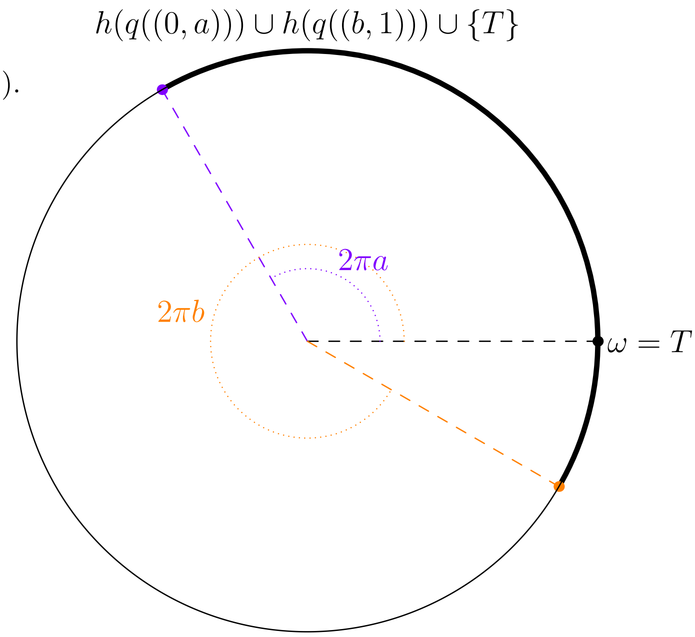
If $\omega = T$ then it must be that $[*] \in U$, since $h^{-1}(\{T\}) = \{[*]\}$, and, since $[*] \in U$, it must be that $\{0,1\} \sqcup \{*\} \subseteq q^{-1}(U)$. Now, by the definition of the quotient topology, since $U \subseteq P$ is open, $q^{-1}(U)$ must be an open subset of $X = [0,1] \sqcup \{*\}$. Now $q^{-1}(U) \subseteq X$ is an open set containing $\{0,1\} \sqcup \{*\}$, so it must contain open neighbourhoods in X of each of 0, 1, and $*$. Hence for some $a, b \in (0,1)$ we must have

$$\begin{aligned} &([0,a) \cup (b,1]) \sqcup \{*\} \subseteq q^{-1}(U) \\ \implies\ & q((0,a)) \cup q((b,1)) \cup \{[*]\} \subseteq q(q^{-1}(U)) \subseteq U \\ \implies\ & h(q((0,a))) \cup h(q((b,1))) \cup \{T\} \subseteq h(U). \end{aligned}$$

4. Without loss of generality we may assume that $a < b$. Then $h(q((0,a))) \cup h(q((b,1))) \cup \{T\}$ is an open arc traversed counterclockwise from

$$(\cos(2\pi b), \sin(2\pi b)) \quad \text{to} \quad (\cos(2\pi a), \sin(2\pi a)).$$

In particular $T = \omega$ is a point on the arc. Thus, in the case where $\omega = T$, we have produced an open arc contained in $h(U)$ which contains ω.



If $\omega \neq T$ then $\omega = h([t])$ for some $0 < t < 1$. This is because

$$h^{-1}(S^1 \backslash \{T\}) = P \backslash \{[*]\} \quad \text{and} \quad q^{-1}(P \backslash \{[*]\}) = (0,1).$$

Since $\omega = h([t])$ and the restriction of the maps q and h in

$$\underbrace{(0,1)}_{\subseteq X} \xrightarrow{q} P \setminus \{[*]\} \xrightarrow{h} S^1 \setminus \{T\}$$

are injective, it must be that $[t] \in U$ and $t \in q^{-1}(U)$. Now, by the definition of the quotient topology the set $q^{-1}(U) \subseteq X$ is open, so if $t \in q^{-1}(U)$ then for some open subinterval $(a,b) \subseteq (0,1)$ we have

$$\begin{aligned} & t \in (a,b) \subseteq q^{-1}(U) \\ \implies & [t] = q(t) \in q((a,b)) \subseteq q(q^{-1}(U)) \subseteq U \\ \implies & \omega = h([t]) \in h(q((a,b))) \subseteq h(U). \end{aligned}$$

At this point we note that $h(q((a,b)))$ is an open arc traversed counterclockwise from $(\cos(2\pi a), \sin(2\pi a))$ to $(\cos(2\pi b), \sin(2\pi b))$. Thus, in the case where $\omega \neq T$, we have produced an open arc contained in $h(U)$ which contains ω.

In the case where $\omega = T$ as well as in the case where $\omega \neq T$, we have produced an open arc contained in $h(U)$ which contains $\omega \in U$. Since $\omega \in h(U)$ was arbitrary, we have shown that $h(U) \subseteq S^1$ is open, and since $U \subseteq P$ was an arbitrary open set, we have shown that $h \colon P \longrightarrow S^1$ is an open map.

Conclusion

Altogether we have shown that there exists a continuous map $h \colon P \longrightarrow S^1$ which is bijective and open. It follows that $P \cong S^1$.

Question 5 (4 marks)

Exercise L7-12.

(a) Prove (a, b) is homeomorphic to $\mathbb{R}$ for any $a < b$.

(b) Prove $\Pi \cong S^1 \times S^1$.

5(a) Firstly, every open bounded interval (a, b) is homeomorphic to the open interval $(-\pi/2, \pi/2)$: Take the map

$$\begin{aligned} g\colon \quad & (a,b) \longrightarrow (-\pi/2, \pi/2) \\ & t \longmapsto -\frac{\pi}{2} + \frac{t-a}{b-a}\pi. \end{aligned}$$

The map g is continuous since it is polynomial. It has an inverse

$$\begin{aligned} g^{-1}\colon \quad & (-\pi/2, \pi/2) \longrightarrow (a,b) \\ & t \longmapsto a + \frac{t+\pi/2}{\pi}(b-a) \end{aligned}$$

which is continuous since it is also polynomial. This shows that $(a, b) \cong (-\pi/2, \pi/2)$ if $a < b$.

To show that $(a, b) \cong \mathbb{R}$ for all $a, b \in \mathbb{R}$ where $a < b$ it now suffices to show that $(-\pi/2, \pi/2) \cong \mathbb{R}$. To show this homeomorphism, define the map

$$\begin{aligned} f\colon \quad & (-\pi/2, \pi/2) \longrightarrow \mathbb{R} \\ & x \longmapsto \tan(x). \end{aligned}$$

That is, let f be the restriction of $\tan(-)$ to $(-\pi/2, \pi/2)$. Since $\tan(-)$ is continuous, the map f is also continuous. Moreover, the range of f is $\mathbb{R}$, and f is strictly increasing, so f is a bijection. Finally, since f is strictly increasing and continuous, the image under f of any open interval $(s, t) \subseteq (-\pi/2, \pi/2)$ is an open interval $(\tan(s), \tan(t)) \subseteq \mathbb{R}$. Thus $f\colon (-\pi/2, \pi/2) \longrightarrow \mathbb{R}$ is a continuous and open bijection, and it follows that $(-\pi/2, \pi/2) \cong \mathbb{R}$.

(b) Recall that the torus Π is defined as follows: Let $C := S^1 \times [0, 1]$ be the cylinder, and define the functions $f\colon S^1 \longrightarrow C$ and $g\colon S^1 \longrightarrow C$ by

$$\begin{aligned} f(x) &:= (x, 0) \quad \forall x \in S^1 \\ g(x) &:= (x, 1) \quad \forall x \in S^1. \end{aligned}$$

We define $\sim$ to be the equivalence relation on C generated by $\{(f(x), g(x)) \mid x \in S^1\}$. Then the torus Π is defined as $\Pi := C/\sim$.

We proceed as follows.

(i) Produce a continuous map $h\colon C \longrightarrow S^1 \times S^1$ using the universal property of the product.

(ii) Produce a continuous map $p\colon \Pi \longrightarrow S^1 \times S^1$ induced by h using the universal property of the quotient space.

(iii) Prove that p is bijective and open.

Producing continuous $h\colon C \longrightarrow S^1 \times S^1$

Define the function $u\colon [0, 1] \longrightarrow S^1$ by

$$u(t) := (\cos(2\pi t), \sin(2\pi t)) \quad \forall t \in [0, 1].$$

5(b) As shown in Exercise L7-10 (Question 4), the map u is continuous. Define a function $h\colon C \longrightarrow S^1 \times S^1$ by

$$h(x,t) := (x, u(t)) \quad \forall x \in S^1 \ \forall t \in [0,1].$$

We use the universal property of the product to show that h is continuous. Observe that

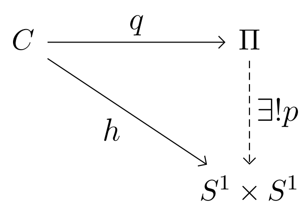
(i) The map $(x,t) \longmapsto x$, which is h projected to the first coordinate, is continuous: The inverse image of an open set $U \subseteq S^1$ is $U \times [0,1]$, which is an open set in C.

(ii) The map $(x,t) \longmapsto u(t)$, which is h projected to the second coordinate, is continuous: The inverse image of an open set $V \subseteq S^1$ is $S^1 \times u^{-1}(V)$. Since $u\colon [0,1] \longrightarrow S^1$ is continuous, $u^{-1}(V)$ is open in $[0,1]$ and $S^1 \times u^{-1}(V)$ is open in C.

It follows by the universal property of the product that the map $h\colon C \longrightarrow S^1 \times S^1$ is continuous.

Producing continuous $p\colon \Pi \longrightarrow S^1 \times S^1$

Our next step is to produce $p\colon \Pi \longrightarrow S^1 \times S^1$ from h using the universal property of the quotient space. Let $q\colon C \longrightarrow \Pi$ be the quotient map.



We wish to check that $h(x_1,t_1) = h(x_2,t_2)$ whenever (x_1,t_1) and (x_2,t_2) are elements of C where $(x_1,t_1) \sim (x_2,t_2)$, with $\sim$ as defined before. Since $\sim$ was the equivalence relation on C generated by $\{(f(x), g(x)) \mid x \in S^1\}$, it suffices to check that $h(f(x)) = h(g(x))$ for every $x \in S^1$. Observe that

$$h(f(x)) = h(x,1) = (x, u(1)) = (x, u(0)) = h(g(x)) \quad \forall x \in S^1,$$

where we have used the fact that $u(1) = (1,0) = u(0)$ in S^1. Thus $h(x_1,t_1) = h(x_2,t_2)$ whenever (x_1,t_1) and (x_2,t_2) are elements of C equivalent under $\sim$. Invoking the universal property of the quotient space, there exists a unique continuous map $p\colon \Pi \longrightarrow S^1 \times S^1$ satisfying $h = p \circ q$ as maps from C to $S^1 \times S^1$.

$p\colon \Pi \longrightarrow S^1 \times S^1$ is bijective

We now turn to showing that the induced map p is bijective. Since $h\colon C \longrightarrow S^1 \times S^1$ is surjective and $h = p \circ q$, the map $p\colon \Pi \longrightarrow S^1 \times S^1$ must also be surjective. Next we show that p is injective. Since the quotient map $q\colon C \longrightarrow \Pi$ is surjective, in order to show that p is injective it is sufficient to show that the following holds:

$$\forall (x_1,t_1),(x_2,t_2) \in C \ \Big(p(q(x_1,t_1)) = p(q(x_2,t_2)) \implies q(x_1,t_1) = q(x_2,t_2)\Big).$$

Since $h = p \circ q$, this is equivalent to

$$\forall (x_1,t_1),(x_2,t_2) \in C \ \Big(h(x_1,t_1) = h(x_2,t_2) \implies q(x_1,t_1) = q(x_2,t_2)\Big).$$

Now, h projected to the first coordinate is the map $(x,t) \longmapsto x$, so if $h(x_1,t_1) = h(x_2,t_2)$ then $x_1 = x_2$. Hence to show injectivity of p it is sufficient to show that

$$\forall x \in S^1 \ \forall t_1,t_2 \in [0,1] \ \Big(h(x,t_1) = h(x,t_2) \implies q(x,t_1) = q(x,t_2)\Big).$$

Let $x \in S^1$ and $t_1, t_2 \in [0,1]$ be arbitrary. Since $h(x,t_1) = (x, u(t_1))$ and $h(x,t_2) = (x, u(t_2))$, if $h(x,t_1) = h(x,t_2)$ then $u(t_1) = u(t_2)$. Recalling that $u(s) = (\cos(2\pi s), \sin(2\pi s))$ for every $s \in [0,1]$, if $u(t_1) = u(t_2)$

5(b) then $t_1 = t_2$ or $\{t_1, t_2\} = \{0, 1\}$. If $t_1 = t_2$ then $q(x, t_1) = q(x, t_2)$. If $\{t_1, t_2\} = \{0, 1\}$ then $\{(x, t_1), (x, t_2)\} = \{f(x), g(x)\}$. That is, if $\{t_1, t_2\} = \{0, 1\}$ then $(x, t_1) \sim (x, t_2)$ and thus by definition $q(x, t_1) = q(x, t_2)$. This establishes injectivity of p. Altogether, we have shown that p is injective and surjective, so p is bijective.

$p\colon \Pi \longrightarrow S^1 \times S^1$ is open

The last step is to show that p is open. Take an arbitrary open set $U \subseteq \Pi$, and take an arbitrary point $(x, t) \in q^{-1}(U)$. We will show that $p(U)$ contains an open neighbourhood of $(p \circ q)(x, t)$. Since (x, t) was an arbitrary point in $q^{-1}(U)$ and $U = q(q^{-1}(U))$, this will be sufficient to show that $p(U)$ is open. Note that $U = q(q^{-1}(U))$ is a consequence of q being surjective.

If $t \in \{0, 1\}$ then since $q^{-1}(U) \subseteq C$ is saturated with respect to $\sim$ we must have that both $(x, 0)$ and $(x, 1)$ are points in $q^{-1}(U)$. Since $q^{-1}(U) \subseteq C$ is open there must exist some open neighbourhood $A \subseteq S^1$ of x and $a, b \in (0, 1)$ such that

$$(x, t) \in A \times ([0, a) \cup (b, 1]) \subseteq q^{-1}(U).$$

We will assume that $a < b$, and there is no loss of generality in doing so.

$$\begin{aligned}
&(x, t) \in A \times ([0, a) \cup (b, 1]) \subseteq q^{-1}(U) \\
&\implies (p \circ q)(x, t) \in (p \circ q)(A \times ([0, a) \cup (b, 1])) \subseteq q(q^{-1}(U)) = p(U)
\end{aligned}$$

Define the point $T := (1, 0) \in S^1$. Recalling that $h(w, s) = (w, u(s))$ for every $(w, s) \in C = S^1 \times [0, 1]$, we have

$$\begin{aligned}
&(p \circ q)(A \times ([0, a) \cup (b, 1])) \\
&= (p \circ q)(A \times \{0, 1\}) \cup (p \circ q)(A \times (0, a)) \cup (p \circ q)(A \times (b, 1)) \\
&= h(A \times \{0, 1\}) \cup h(A \times (0, a)) \cup h(A \times (b, 1)) \\
&= (A \times \{T\}) \cup (A \times u((0, a))) \cup (A \times u((b, 1))) \\
&= A \times (u((b, 1)) \cup \{T\} \cup u((0, a))).
\end{aligned}$$

Since $a < b$, the set $u((b, 1)) \cup \{T\} \cup u((0, a)) \subseteq S^1$ is an open arc, and since $A \subseteq S^1$ is open, it follows that

$$A \times (u((b, 1)) \cup \{T\} \cup u((0, a))) \subseteq S^1 \times S^1$$

is an open set. Thus if $t \in \{0, 1\}$ then we have identified an open neighbourhood of $(p \circ q)(x, t)$ in $p(U)$.

If $0 < t < 1$ then since $q^{-1}(U) \subseteq C$ is open there must exist some open neighbourhood $A \subseteq S^1$ of x and a subinterval $(a, b) \subseteq (0, 1)$ such that

$$\begin{aligned}
&(x, t) \in A \times (a, b) \subseteq q^{-1}(U) \\
&\implies (p \circ q)(x, t) \in (p \circ q)(A \times (a, b)) \subseteq (p \circ q)(q^{-1}(U)) = p(U).
\end{aligned}$$

Now $(p \circ q)(A \times (a, b)) = h(A \times (a, b)) = A \times u((a, b))$. As $A \subseteq S^1$ is open and $u((a, b)) \subseteq S^1$ is an open arc, it follows that $A \times u((a, b)) \subseteq S^1 \times S^1$ is open. Thus if $0 < t < 1$ then we have identified an open neighbourhood of $(p \circ q)(x, t)$ in $p(U)$.

In the case where $t \in \{0, 1\}$ as well as the case where $0 < t < 1$, we have shown that $(p \circ q)(x, t)$ has an open neighbourhood contained in $p(U)$. Since (x, t) was an arbitrary point in $q^{-1}(U)$, this shows that $p(U)$ is open, and since $U \subseteq \Pi$ was an arbitrary open set, we have shown that $p\colon \Pi \longrightarrow S^1 \times S^1$ is open.

Conclusion

We have produced a map $p\colon \Pi \longrightarrow S^1 \times S^1$ which is continuous, bijective, and open. It follows that $\Pi \cong S^1 \times S^1$.

Question 6 (5 marks)

Exercise L7-19. Write D^n/S^{n-1} for the quotient space $D^n/\sim$ where $\sim$ is the smallest equivalence relation with $x \sim y$ for all $x, y \in S^{n-1} \subseteq D^n$.

(a) Prove $D^2/S^1 \cong S^2$.

(b) Prove $D^n/S^{n-1} \cong S^n$ for $n > 2$.

(c) Prove S^n is a finite CW-complex by attaching a single n-cell to a single 0-cell (i.e. all intermediate stages have Λ empty).

Note. For this exercise *only*, you may use that a continuous bijection from a compact space to a Hausdorff space is a homeomorphism.

By Corollary L10-4, for every $n \in \mathbb{N}$, the n-disk $D^n \subseteq \mathbb{R}^n$ and the n-sphere $S^n \subseteq \mathbb{R}^{n+1}$ are compact. By Lemma L10-1, quotient spaces of compact topological spaces are compact, so D^n/S^{n-1} is compact for every $n \in \mathbb{N}$.

6(a) This is a special case of the next part. The argument provided there holds in full generality for integers $n \geqslant 2$.

(b) Fix an integer $n \geqslant 2$. We will define a continuous map $f\colon D^n \longrightarrow S^n$ which sends an $(n-1)$-sphere (centred at the origin) of radius $r \in [0,1]$ to the intersection of a hyperplane in $\mathbb{R}^{n+1}$ and S^n. Our definition of f will be such that the image of S^{n-1} under f is a single point in S^n. We will then use the universal property of the quotient space to induce a continuous map $p\colon D^n/S^{n-1} \longrightarrow S^n$, and we will show that p is bijective. Finally, we will use the result that a continuous bijection from a compact space to a Hausdorff space is a homeomorphism to conclude that $D^n/S^{n-1} \cong S^n$.

Define $f\colon D^n \longrightarrow S^n$ by

$$\begin{aligned} f\colon \quad & D^n \longrightarrow S^n \\ & \boldsymbol{a} \longmapsto \left(\tfrac{\boldsymbol{a}\sin(\pi\|\boldsymbol{a}\|)}{\|\boldsymbol{a}\|}, \cos(\pi\|\boldsymbol{a}\|)\right) \\ & \boldsymbol{0} \longmapsto (\boldsymbol{0}, 1) \in \mathbb{R}^{n+1}, \end{aligned}$$

where the $\boldsymbol{0}$ in $(\boldsymbol{0}, 1)$ represents the first n coordinates of $f(\boldsymbol{0})$ all being 0. On $D^n \setminus \{0\}$, we have

$$f(a_1, a_2, \ldots, a_n) = \left(\frac{a_1}{\|\boldsymbol{a}\|}\sin(\pi\|\boldsymbol{a}\|),\ \frac{a_2}{\|\boldsymbol{a}\|}\sin(\pi\|\boldsymbol{a}\|),\ \ldots,\ \frac{a_n}{\|\boldsymbol{a}\|}\sin(\pi\|\boldsymbol{a}\|),\ \cos(\pi\|\boldsymbol{a}\|)\right) \quad \forall \boldsymbol{a} \in D^n \setminus \{\boldsymbol{0}\}.$$

Note that our separate specification of $f(\boldsymbol{0})$ makes f continuous at $\boldsymbol{0} \in D^n$ — we will use this fact when we show that f is continuous. Let us verify that the map is well-defined: Since $\|(\boldsymbol{0}, 1)\| = 1$, we see that $f(\boldsymbol{0})$ is in S^n. We also have

$$\begin{aligned} \left\| \left(\frac{\boldsymbol{a}\sin(\pi\|\boldsymbol{a}\|)}{\|\boldsymbol{a}\|}, \cos(\pi\|\boldsymbol{a}\|)\right) \right\| &= \sqrt{\left\|\frac{\boldsymbol{a}\sin(\pi\|\boldsymbol{a}\|)}{\|\boldsymbol{a}\|}\right\|^2 + \cos^2(\pi\|\boldsymbol{a}\|)} \\ &= \sqrt{\sin^2(\pi\|\boldsymbol{a}\|) + \cos^2(\pi\|\boldsymbol{a}\|)} = 1. \end{aligned}$$

Thus the image of every point in D^n under f is indeed in S^n. Let us validate our previous description of f

6(b) as sending $(n-1)$-spheres to $(n-1)$-spheres: If $r \in [0,1]$ then

$$\begin{aligned} f(rS^{n-1}) &= \{f(r\boldsymbol{b}) \mid \boldsymbol{b} \in S^{n-1}\} \\ &= \left\{ \left(\frac{rb_1}{\|r\boldsymbol{b}\|} \sin(\pi\|r\boldsymbol{b}\|),\ \frac{rb_2}{\|r\boldsymbol{b}\|} \sin(\pi\|r\boldsymbol{b}\|),\ \ldots,\ \frac{rb_n}{\|r\boldsymbol{b}\|} \sin(\pi\|r\boldsymbol{b}\|),\ \cos(\pi\|r\boldsymbol{b}\|) \right) \;\middle|\; \boldsymbol{b} \in S^{n-1} \right\} \\ &= \{(b_1 \sin(\pi r), b_2 \sin(\pi r), \ldots, b_n \sin(\pi r), \cos(\pi r)) \mid \boldsymbol{b} \in S^{n-1}\} \\ &= \{(\sin(\pi r)\boldsymbol{b}, \cos(\pi r)) \mid \boldsymbol{b} \in S^{n-1}\}, \end{aligned}$$

which is an $(n-1)$-sphere contained in $S^n \subseteq \mathbb{R}^{n+1}$ with radius $\sin(\pi r)$ and centre $(\boldsymbol{0}, \cos(\pi r)) \in \mathbb{R}^{n+1}$.

$f\colon D^n \longrightarrow S^n$ is continuous

We now show that f is continuous. By the universal property of the product, in order to show that f is continuous, it is equivalent to check that each of the following maps is continuous:

$$(\pi_i \circ f)(\boldsymbol{a}) = \frac{a_i}{\|\boldsymbol{a}\|} \sin(\pi\|\boldsymbol{a}\|), \quad i \in \{1, 2, \ldots, n\}.$$
$$(\pi_{n+1} \circ f)(\boldsymbol{a}) = \cos(\pi\|\boldsymbol{a}\|).$$

We will adopt the convention that the map from $\mathbb{R}_{\geqslant 0}$ to $\mathbb{R}$ given by $t \longmapsto \sin(\pi t)/t$ takes 0 to π. As a consequence, $t \longmapsto \sin(\pi t)/t$ will be continuous on all of $\mathbb{R}_{\geqslant 0}$.

First note that the map $\boldsymbol{a} \longmapsto \|\boldsymbol{a}\|$ is continuous: If $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ and $\varepsilon > 0$ then

$$\begin{aligned} \|\boldsymbol{y} - \boldsymbol{x}\| < \varepsilon \quad &\Longrightarrow \quad \|\boldsymbol{x}\| + \|\boldsymbol{y} - \boldsymbol{x}\| < \|\boldsymbol{x}\| + \varepsilon \\ &\Longrightarrow \quad \|\boldsymbol{y}\| < \|\boldsymbol{x}\| + \varepsilon \quad \text{by the triangle inequality.} \\ \|\boldsymbol{y} - \boldsymbol{x}\| < \varepsilon \quad &\Longrightarrow \quad \|\boldsymbol{y}\| + \|\boldsymbol{x} - \boldsymbol{y}\| < \|\boldsymbol{y}\| + \varepsilon \\ &\Longrightarrow \quad \|\boldsymbol{x}\| < \|\boldsymbol{y}\| + \varepsilon \quad \text{by the triangle inequality.} \end{aligned}$$

That is, if $\boldsymbol{y} \in B_\varepsilon(\boldsymbol{x})$ then $|\|\boldsymbol{x}\| - \|\boldsymbol{y}\|| < \varepsilon$. (This is the ε–δ definition of continuity for the map $\boldsymbol{a} \longmapsto \|\boldsymbol{a}\|$.) Next, since the map from $\mathbb{R}_{\geqslant 0}$ to $\mathbb{R}$ given by $t \longmapsto \sin(\pi t)/t$ is continuous, it follows that the composition

$$\boldsymbol{a} \longmapsto \|\boldsymbol{a}\| \longmapsto \frac{\sin(\pi\|\boldsymbol{a}\|)}{\|\boldsymbol{a}\|}$$

is a continuous map from D^n to $\mathbb{R}$. Furthermore, for every $i \in \{1, 2, \ldots, n\}$, the projection map from D^n to $\mathbb{R}$ given by $\boldsymbol{a} \longmapsto a_i$ is continuous.

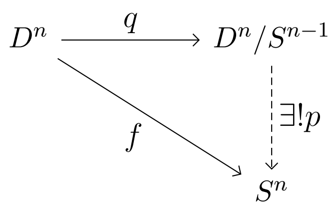
Now, for $i \in \{1, 2, \ldots, n\}$, each map $\pi_i \circ f\colon D^n \longrightarrow \mathbb{R}$ is given by

$$(\pi_i \circ f)(\boldsymbol{a}) = \frac{a_i}{\|\boldsymbol{a}\|} \sin(\pi\|\boldsymbol{a}\|),$$

which is a pointwise product of the two continuous maps $\boldsymbol{a} \longmapsto \sin(\pi\|\boldsymbol{a}\|)/\|\boldsymbol{a}\|$ and $\boldsymbol{a} \longmapsto a_i$, so $\pi_i \circ f$ itself must be a continuous map. This holds true for every $i \in \{1, 2, \ldots, n\}$.

It remains to check that $\pi_{n+1} \circ f\colon D^n \longrightarrow \mathbb{R}$ is continuous. Observe that $\pi_{n+1} \circ f$ is a composition of $\boldsymbol{a} \longmapsto \|\boldsymbol{a}\|$ and $t \longmapsto \cos(\pi t)$, both of which are continuous maps. Hence $\pi_{n+1} \circ f$ is also continuous. Since $\pi_i \circ f$ is continuous for every $i \in \{1, 2, \ldots, n, n+1\}$, the map $f\colon D^n \longrightarrow S^n$ is continuous by the universal property of the product.

Inducing $p\colon D^n/S^{n-1} \longrightarrow S^n$ from $f\colon D^n \longrightarrow S^n$



6(b) In order to use f to induce a continuous map $p\colon D^n/S^{n-1} \longrightarrow S^n$, we must also check that f is constant on $S^{n-1} \subseteq D^n$. Indeed, if $\boldsymbol{a} \in \mathbb{R}^n$ and $\|\boldsymbol{a}\| = 1$ then

$$f(\boldsymbol{a}) = (\boldsymbol{a}\sin(\pi), \cos(\pi)) = (\boldsymbol{0}, -1) \in \mathbb{R}^{n+1},$$

where the $\boldsymbol{0}$ represents the first n coordinates of $f(\boldsymbol{a})$ all being 0. Thus f is constant on S^{n-1}. Letting $q\colon D^n \longrightarrow D^n/S^{n-1}$ be the quotient map, by the universal property of the quotient space there exists a unique map $p\colon D^n/S^{n-1} \longrightarrow S^n$ such that $f = p \circ q$.

$p\colon D^n/S^{n-1} \longrightarrow S^n$ is bijective

Our next step is to show that $p\colon D^n/S^{n-1} \longrightarrow S^n$ is bijective. Let us first consider surjectivity. Since $f = p \circ q$, in order to show that p is surjective, it is sufficient to show that f is surjective. For each $z \in [-1,1]$, define $H_z \coloneqq \{\boldsymbol{b} \in S^n \mid \pi_{n+1}(\boldsymbol{b}) = z\}$. We can see that $\{H_z \mid z \in [-1,1]\}$ is a partition of S^n, so in order to show that f is surjective, it suffices to show that H_z is inside the range of f for every $z \in [-1,1]$. Let us look more closely at the points inside H_z.

$$\begin{aligned} H_z &= \{\boldsymbol{b} \in S^n \mid \pi_{n+1}(\boldsymbol{b}) = z\} \\ &= \{(b_1, b_2, \ldots, b_n, z) \in \mathbb{R}^{n+1} \mid b_1^2 + b_2^2 + \cdots + b_n^2 + z^2 = 1\}. \end{aligned}$$

Since the map from $[0,1]$ to $[-1,1]$ given by $t \longmapsto \cos(\pi t)$ is bijective, it is equivalent to consider $H_{\cos(\pi y)}$ for $0 \leqslant y \leqslant 1$.

$$\begin{aligned} H_{\cos(\pi y)} &= \{(b_1, b_2, \ldots, b_n, \cos(\pi y)) \in \mathbb{R}^{n+1} \mid b_1^2 + b_2^2 + \cdots + b_n^2 + \cos^2(\pi y) = 1\} \\ &= \{(b_1, b_2, \ldots, b_n, \cos(\pi y)) \in \mathbb{R}^{n+1} \mid b_1^2 + b_2^2 + \cdots + b_n^2 = \sin^2(\pi y)\} \\ &= \{(c_1\sin(\pi y), c_2\sin(\pi y), \ldots, c_n\sin(\pi y), \cos(\pi y)) \mid \boldsymbol{c} \in S^{n-1}\}. \end{aligned}$$

At this point we observe that $H_{\cos(\pi y)}$ is precisely the image under f of an $(n-1)$-sphere (contained in D^n) centred at the origin with radius y. That is,

$$H_{\cos(\pi y)} = \{f(\boldsymbol{a}) \mid \boldsymbol{a} \in \mathbb{R}^n,\ \|\boldsymbol{a}\| = y \in [0,1]\} = f(yS^{n-1}).$$

Since this holds for every $y \in [0,1]$ and $S^n = \bigcup_{y\in[0,1]} H_{\cos(\pi y)}$, it follows that f is surjective, and therefore p is surjective also.

Next we wish to show that p is injective. Since $q\colon D^n \longrightarrow D^n/S^{n-1}$ is surjective it is enough to check that the following holds:

$$\forall \boldsymbol{a}, \boldsymbol{b} \in D^n \ \Big(p(q(\boldsymbol{a})) = p(q(\boldsymbol{b})) \implies q(\boldsymbol{a}) = q(\boldsymbol{b})\Big).$$

Noting that $p \circ q = f$, the above statement is equivalent to

$$\forall \boldsymbol{a}, \boldsymbol{b} \in D^n \ \Big(f(\boldsymbol{a}) = f(\boldsymbol{b}) \implies q(\boldsymbol{a}) = q(\boldsymbol{b})\Big).$$

Suppose $\boldsymbol{a}, \boldsymbol{b} \in D^n$ are such that $f(\boldsymbol{a}) = f(\boldsymbol{b})$. Then, recalling the definition of f, we have

$$\begin{aligned} &\left(\frac{a_1}{\|\boldsymbol{a}\|}\sin(\pi\|\boldsymbol{a}\|),\ \frac{a_2}{\|\boldsymbol{a}\|}\sin(\pi\|\boldsymbol{a}\|),\ \ldots,\ \frac{a_n}{\|\boldsymbol{a}\|}\sin(\pi\|\boldsymbol{a}\|),\ \cos(\pi\|\boldsymbol{a}\|)\right) \\ &= \left(\frac{b_1}{\|\boldsymbol{b}\|}\sin(\pi\|\boldsymbol{b}\|),\ \frac{b_2}{\|\boldsymbol{b}\|}\sin(\pi\|\boldsymbol{b}\|),\ \ldots,\ \frac{b_n}{\|\boldsymbol{b}\|}\sin(\pi\|\boldsymbol{b}\|),\ \cos(\pi\|\boldsymbol{b}\|)\right) \end{aligned}$$

Since $\|\boldsymbol{a}\|, \|\boldsymbol{b}\| \in [0,1]$ and $\cos(\pi\|\boldsymbol{a}\|) = \cos(\pi\|\boldsymbol{b}\|)$, we must have $\|\boldsymbol{a}\| = \|\boldsymbol{b}\|$. Hence we also have $\sin(\pi\|\boldsymbol{a}\|)/\|\boldsymbol{a}\| = \sin(\pi\|\boldsymbol{b}\|)/\|\boldsymbol{b}\|$. If $0 \leqslant \|\boldsymbol{a}\| < 1$, then $\sin(\pi\|\boldsymbol{a}\|)/\|\boldsymbol{a}\| \neq 0$ (recall that we consider 0 to map to π under $t \longmapsto \sin(\pi t)/t$), so

$$\frac{a_i}{\|\boldsymbol{a}\|}\sin(\pi\|\boldsymbol{a}\|) = \frac{b_i}{\|\boldsymbol{b}\|}\sin(\pi\|\boldsymbol{b}\|) \qquad \implies \qquad a_i = b_i.$$

6(b) This holds for every $i \in \{1, 2, \ldots, n\}$, so $\boldsymbol{a} = \boldsymbol{b}$ and therefore $q(\boldsymbol{a}) = q(\boldsymbol{b})$. If $\|\boldsymbol{a}\| = 1$ then $\boldsymbol{a}$ and $\boldsymbol{b}$ are both points in S^{n-1}, so certainly $q(\boldsymbol{a}) = q(\boldsymbol{b})$. Thus we have shown that if $\boldsymbol{a}, \boldsymbol{b} \in D^n$ are such that $f(\boldsymbol{a}) = f(\boldsymbol{b})$ then necessarily $q(\boldsymbol{a}) = q(\boldsymbol{b})$. This establishes injectivity of p. Since p is both injective and surjective, p is bijective.

Conclusion

We have now exhibited a continuous bijection p from D^n/S^{n-1} to S^n. Now, D^n/S^{n-1} is a compact space as reasoned at the beginning of the question. Since S^n is a subspace of the Hausdorff space $\mathbb{R}^{n+1}$, we know that S^n itself is Hausdorff. Finally, using the fact that a continuous bijection from a compact space to a Hausdorff space is a homeomorphism, in light of the continuous bijection $p\colon D^n/S^{n-1} \longrightarrow S^n$, it must be that $D^n/S^{n-1} \cong S^n$.

(c) Begin with $X_0 \coloneqq \{*\}$, a single 0-cell. Set $X_i \coloneqq X_0$ for $i \in \{1, 2, \ldots, n-1\}$. That is, for every $i \in \{1, 2, \ldots, n-1\}$, at stage i of attaching i-cells to X_{i-1}, we choose not to attach any ($\Lambda_i = \emptyset$). At stage n, we attach a single n-cell according to the pushout

$$\begin{array}{ccc} S^{n-1} & \xrightarrow{\ f\ } & X_{n-1} = \{*\} \\ \big\downarrow & & \big\downarrow \\ D^n & \longrightarrow & X_n \coloneqq (\{*\} \sqcup D^n)/\approx. \end{array}$$

The map f sends every point in S^{n-1} to $* \in X_{n-1}$. This map is continuous since the open sets in X_{n-1} are $\emptyset$ and $\{*\}$ which have respective preimages under f of $\emptyset$ and S^{n-1}, both of which are open in S^{n-1}. The equivalence relation $\approx$ on $\{*\} \sqcup D^n$ is generated by

$$\{(x, y) \mid x, y \in \{*\} \sqcup S^{n-1}\}.$$

If we can show that $S^n \cong X_n$, then we are done. From the previous part, we know that $S^n \cong D^n/S^{n-1}$, so we only have to show that $D^n/S^{n-1} \cong X_n = (\{*\} \sqcup D^n)/\approx$. Intuitively this seems very obvious, since the equivalence relation $\approx$ on $\{*\} \sqcup D^n$ induces the following partition on $\{*\} \sqcup D^n$:

$$\{\{*\} \sqcup S^{n-1}\} \cup \{\{x\} \mid x \in D^n \setminus S^{n-1}\}.$$

Meanwhile the equivalence relation $\sim$ on D^n generated by declaring everything in S^{n-1} to be equivalent is

$$\{S^{n-1}\} \cup \{\{x\} \mid x \in D^n \setminus S^{n-1}\}.$$

Fix a point $T_n \in S^{n-1}$. Define the maps $f\colon D^n/S^{n-1} \longrightarrow X_n$ and $g\colon X_n \longrightarrow D^n/S^{n-1}$ by

$$\begin{aligned} f([x]) &= [x], \quad x \in D^n. \\ g([x]) &= [x], \quad x \in D^n \setminus S^{n-1}, \\ g([*]) &= [T_n]. \end{aligned}$$

It can be checked that these maps are well-defined by referring to the explicit partitions induced by each equivalence relation on $\{*\} \sqcup D^n$ and D^n. We will show that f and g are inverse to each other and that each is continuous, thereby showing $D^n/S^{n-1} \cong X_n$.

$f\colon D^n/S^{n-1} \longrightarrow X_n$ and $g\colon X_n \longrightarrow D^n/S^{n-1}$ are inverses

Note that $(g \circ f)([x]) = [x]$ for every $x \in D^n$. Also

$$\begin{aligned} (f \circ g)([x]) &= [x], && x \in D^n \setminus S^{n-1}, \\ (f \circ g)([x]) &= (f \circ g)([*]) = f([T_n]) \\ &= [T_n] = [x], && x \in \{*\} \sqcup S^{n-1}, \end{aligned}$$

6(c) so that $(f \circ g)([x]) = [x]$ for every $x \in \{*\} \sqcup D^n$. This means that $f: D^n/S^{n-1} \longrightarrow X_n$ and $g: X_n \longrightarrow D^n/S^{n-1}$ are inverse to each other.

$f: D^n/S^{n-1} \longrightarrow X_n$ and $g: X_n \longrightarrow D^n/S^{n-1}$ are both continuous

Now we show continuity of f and g. Since they are inverse to each other, it is equivalent to show that f and g are each open. Let $q: D^n \longrightarrow D^n/S^{n-1}$ and $w: \{*\} \sqcup D^n \longrightarrow X_n$ be the quotient maps. Let $B^n := D^n \setminus S^{n-1}$ denote the open unit ball in $\mathbb{R}^n$.

The important underlying argument in the following (opaque and uninteresting) discussion is that open sets in D^n/S^{n-1} and X_n are images under q and w of saturated open sets in D^n and $\{*\} \sqcup D^n$ respectively. By the nature of the equivalence relations on each of D^n and $\{*\} \sqcup D^n$, saturated open sets in one can be changed into saturated open sets of the other by adding or removing $*$ as appropriate. More specifically, if $U_0 \subseteq D^n$ and $V_0 \subseteq \{*\} \sqcup D^n$ are open sets that are saturated in D^n and $\{*\} \sqcup D^n$ respectively, then defining

$$U_1 := \begin{cases} \{*\}, & S^{n-1} \subseteq U_0, \\ \emptyset, & S^{n-1} \cap U_0 = \emptyset, \end{cases} \quad \text{and} \quad V_1 := \begin{cases} \{*\}, & * \in V_0, \\ \emptyset, & (\{*\} \sqcup S^{n-1}) \cap V_0 = \emptyset, \end{cases}$$

we can see that $U_1 \sqcup U_0$ is a saturated open set in $\{*\} \sqcup D^n$, while $V_0 \setminus V_1$ is a saturated open set in D^n by the definition of the disjoint union topology.

Below is the (opaque and uninteresting) formal argument for why f and g are open.

We first show that f is open. Let $U \subseteq D^n/S^{n-1}$ be an open subset. We will show that $f(U) \subseteq X_n$ is open. First observe that $f(U) = w(q^{-1}(U))$ since f sends $[x] \in D^n/S^{n-1}$ to $[x] \in X_n$. We know that $q^{-1}(U)$ is open when regarded as a subset of $\{*\} \sqcup D^n$ (since it is open as a subset of D^n). Thus, to show that $f(U)$ is open, it suffices to show that $q^{-1}(U)$ is saturated as a subset of $\{*\} \sqcup D^n$. However, this is not always true, since $* \notin q^{-1}(U)$ but $q^{-1}(U)$ may contain elements of S^{n-1}. There is a resolution by splitting into cases: one where $[T_n] \notin U$ and the other where $[T_n] \in U$.

Now, if $[T_n] \notin U$ then $q^{-1}(U) \subseteq B^n$, and indeed this must be saturated in $\{*\} \sqcup D^n$ in light of the induced partition on $\{*\} \sqcup D^n$. That is, if $[T_n] \notin U$ then $f(U) \subseteq X_n$ is open. Otherwise, if $[T_n] \in U$ then $S^{n-1} \subseteq q^{-1}(U)$, and we can in fact write $f(U) = w(\{*\} \sqcup q^{-1}(U))$. Since $\{*\} \sqcup q^{-1}(U)$ is an open subset of $\{*\} \sqcup D^n$ which contains $\{*\} \sqcup S^{n-1}$, the open set $\{*\} \sqcup q^{-1}(U)$ is in fact saturated, so $f(U)$ is open.

Next, we show that g is open. Let $V \subseteq X_n$ be an open subset. We want to show that $g(V) \subseteq D^n/S^{n-1}$ is open. If $[*] \notin V$ then $w^{-1}(V) \subseteq B^n$ inside $\{*\} \sqcup D^n$. This means that if $[*] \notin V$ then $g(V) = q(w^{-1}(V))$, because $g([x]) = [x] \in D^n/S^{n-1}$ if $x \in B^n$. It now suffices to show that $w^{-1}(V)$ is saturated in D^n. Indeed, $w^{-1}(V) \subseteq B^n$, so in light of the induced partition on D^n the set $w^{-1}(V)$ is saturated. If $[*] \in V$ then $\{*\} \sqcup S^{n-1} \in w^{-1}(V)$. Removing $*$ from $w^{-1}(V)$, the set $w^{-1}(V) \setminus \{*\}$ is an open subset of D^n (by the definition of the disjoint union topology), and we still have $g(V) = q(w^{-1}(V) \setminus \{*\})$. Since $S^{n-1} \subseteq w^{-1}(V) \setminus \{*\}$ inside D^n, the set $w^{-1}(V) \setminus \{*\}$ is saturated inside D^n, so $g(V)$ is open.

Conclusion

Since $f: D^n/S^{n-1} \longrightarrow X_n$ and $g: X_n \longrightarrow D^n/S^{n-1}$ are inverse maps to each other and each is open, it follows that $D^n/S^{n-1} \cong X_n$. Hence S^n is indeed a finite CW-complex obtained by attaching a single n-cell to a single 0-cell.

Question 7 (2 marks)

Exercise L8-5. Prove that if $X \subseteq \mathbb{R}$ is *not* sequentially compact, there exists a continuous function $f\colon X \longrightarrow \mathbb{R}$ which is not bounded (i.e. $f(X) \subseteq \mathbb{R}$ is not bounded).

Recall the Bolzano–Weierstrass theorem (Theorem L8-2), which says that a subset $K \subseteq \mathbb{R}$ is closed and bounded if and only if K is sequentially compact. This means that if $X \subseteq \mathbb{R}$ is not sequentially compact then X is nonempty and must be (i) not bounded; or (ii) not closed.

If X is not bounded then taking $f = \mathrm{id}_X$ (which is continuous) we have $f(X) = X \subseteq \mathbb{R}$ which is not bounded. This proves the claim in the case that X is not bounded.

Suppose then that X is not closed. Then by Lemma L8-1 the set X has an adherent point in $\mathbb{R} \setminus X$. Let $c \in \mathbb{R} \setminus X$ be an adherent point of X, and define $f\colon X \longrightarrow \mathbb{R}$ by

$$f(x) = \frac{1}{|x-c|}, \quad x \in X.$$

We will show that the map f as defined above is continuous by showing that the preimage of real bounded open intervals under f is open. This is enough to show continuity since the bounded open intervals form a basis for the topology on $\mathbb{R}$. If $(a,b) \subseteq \mathbb{R}$ is an open interval (where $a < b$) we may compute $f^{-1}((a,b))$ as follows:

(a) If $0 \in [b,\infty)$ (that is, $a < b \leqslant 0$) then since f is positive we have $f^{-1}((a,b)) = \emptyset$, which is an open subset of X.

(b) If $0 \in [a,b)$ (that is, $a \leqslant 0 < b$) then since f is positive we have $f^{-1}((a,b)) = f^{-1}((0,b))$. For $r \in \mathbb{R}$ we have

$$\frac{1}{|r-c|} \in (0,b) \iff |r-c| > \frac{1}{b} \iff r \in \left(-\infty, c-\frac{1}{b}\right) \cup \left(c+\frac{1}{b}, \infty\right).$$

Hence

$$f^{-1}((a,b)) = f^{-1}((0,b)) = X \cap \left(\left(-\infty, c-\frac{1}{b}\right) \cup \left(c+\frac{1}{b}, \infty\right)\right),$$

which is an open subset of X under the subspace topology.

(c) If $0 \in (-\infty,a)$ (that is, $0 < a < b$) then for $r \in \mathbb{R}$ we have

$$\frac{1}{|r-c|} \in (a,b) \iff \frac{1}{b} < |r-c| < \frac{1}{a} \iff r \in \left(c-\frac{1}{a}, c-\frac{1}{b}\right) \cup \left(c+\frac{1}{b}, c+\frac{1}{a}\right).$$

Hence

$$f^{-1}((a,b)) = X \cap \left(\left(c-\frac{1}{a}, c-\frac{1}{b}\right) \cup \left(c+\frac{1}{b}, c+\frac{1}{a}\right)\right),$$

which is an open subset of X under the subspace topology.

Having established that f is continuous, it remains to show that $f(X)$ is not bounded. Since $c \in \mathbb{R} \setminus X$ is an adherent point of X, for every $n \in \mathbb{N}$ there exists $x_n \in X$ such that $|x_n - c| < 1/n$. Having defined the sequence $(x_n)_{n\in\mathbb{N}}$ in X in this manner, observe that

$$f(x_n) = \frac{1}{|x_n - c|} > n \quad \forall n \in \mathbb{N}.$$

This shows that $f(X) \subseteq \mathbb{R}$ is not bounded, which proves the claim in the case that X not closed.

Question 8 (1 mark)

Exercise L9-5. Every closed subspace of a compact topological space is compact.

Let $(X, \mathcal{T})$ be a compact topological space, and let $K \subseteq X$ be closed. We wish to show that K is a compact subspace.

Let $(U_i)_{i \in I}$ be an open cover of K. That is, for each $i \in I$, the set $U_i \subseteq X$ is open and $K \subseteq \bigcup_{i \in I} U_i$. We wish to produce a finite subset $\{i_k\}_{k=1}^n \subseteq I$ such that $K \subseteq \bigcup_{k=1}^n U_{i_k}$.

Let $V := X \setminus K$. Since K is closed in X, we know V is open in X. Since $K \subseteq \bigcup_{i \in I} U_i$, we can see that $V \cup \bigcup_{i \in I} U_i$ is an open cover of X. By compactness of X, the open cover $V \cup \bigcup_{i \in I} U_i$ contains a finite subcover of X. That is, for some finite subset $\{i_k\}_{k=1}^n \subseteq I$ we have

$$X \subseteq V \cup U_{i_1} \cup U_{i_2} \cup \cdots \cup U_{i_n}.$$

(We can choose to include V in the subcover without loss of generality.) Since $K \subseteq X$ this means that

$$K \subseteq V \cup U_{i_1} \cup U_{i_2} \cup \cdots \cup U_{i_n}.$$

Since K and V are disjoint, we can further say that

$$K \subseteq U_{i_1} \cup U_{i_2} \cup \cdots \cup U_{i_n}.$$

Beginning with an arbitrary open cover $(U_i)_{i \in I}$ of K, we have produced a finite subcover $(U_{i_k})_{k=1}^n$ for K. This shows that K is compact.