# Assignment 2

**Jacob Cumming** jacob.cumming@student.unimelb.edu.au
995682

## Exercise L9-7

Let $(X, d)$ be a metric space and $K \subseteq X$ a nonempty compact subset and $x \in X \backslash K$. Set

$$\lambda_x := \inf\{d(k, x) | k \in K\}$$

Prove that there exists $k_o \in K$ with

$$d(k_0, x) = \lambda_x$$

Answer

For any fixed $y$, let $\delta : X \to \mathbb{R}, \quad x \mapsto d(y, x)$. Let $\mathbb{R} \ni U = B_\epsilon(d(x, y)) \in \mathbb{R}$. Let $V = B_\epsilon(x) \in X$. Then for $v \in V$ we have

$$\begin{aligned} \delta(v) &< \delta(x) + \epsilon \\ \delta(x) &< \delta(v) + \epsilon \\ \implies \delta(x) - \epsilon &< \delta(v) < \delta(x) + \epsilon \\ v &\in V \subseteq \delta^{-1}(U) \end{aligned}$$

and so by Lemma L6-4, $\delta$ is a continuous function. Now, restricting $\delta$ to $K$, by Corollary 9.4, since K is compact, there exists $k_0, d \in K$ such that $\delta(k_0) \leq \delta(k) \leq \delta(d) \quad \forall\, k \in K \implies \delta(k_0) = d(k_0, x) = \lambda_x$

## Exercise L11-9

Prove that any compact Hausdorff space is normal (Hint: We use the proof of Lemma 11-5)

Answer

Given a compact Hausdorff space $X$ take any two closed sets say $K, L$. Given $l \in L$ choose for each $k \in K$ a pair of disjoint open sets $U_k, V_k$ with $k \in U_k$ and $l \in V_k$. The $\{U_k\}_{k \in K}$ cover $K$, and since it is compact, finitely many, say $\{U_{k_1}, \ldots, U_{k_n}\}$ will do.

So for every $l \in L$ we have a pair of dijoint open sets $U_l = U_{k_1} \cup \cdots \cup U_{k_n}$ and $V_l = V_{k_1} \cap \cdots \cap V_{k_n}$, which is open since it is a finite intersection of open sets. Clearly $K \subset U_l$ for all $l$, and the $\{V_l\}_{l \in L}$ covers $L$. Since it is compact, finitely many, say $\{V_{l_1}, \ldots, V_{l_n}\}$ will do. Consider the open sets $V = V_{l_1} \cup \cdots \cup V_{l_n}$, and $U = U_{l_1} \cap \cdots \cap U_{l_n}$. $K \subset U$ and $L \subset V$. But $U \cap V = \emptyset$ by construction. Therefore $X$ is normal.

## Exercise L12-3

(i) Prove that if $\{\mathcal{T}_i\}_{i\in I}$ is a collection of topologies on a single set $X$ that $\bigcap_{i\in I}\mathcal{T}_i$ is a topology on $X$.

(ii) With the above notation, prove that $U \in \langle\mathcal{S}\rangle$ if and only if $U$ can be written as a union of sets, each of which is a finite intersection of elements of $\mathcal{S}$

(iii) If $f : X \to Y$ is a function and $\mathcal{S}$ is a sub-basis for the topology on Y, then $f$ is continuous iff. $f^{-1}(U) \subseteq X$ is open for every $U \in \mathcal{S}$.

Answer

(i) Let $\mathcal{T} = \bigcap_{i\in I}\mathcal{T}_i$. $X, \emptyset \in \mathcal{T}_i$ by definition of a topology, so $X, \emptyset \in \bigcap_{i\in I}\mathcal{T}_i$.

For the second condition, take any two $U, V \in \mathcal{T}$. $U, V$ are in all $\mathcal{T}_i$, so $U \cap V \in \mathcal{T}_i$, and hence in $\mathcal{T}$

For the final condition, take $\{U_i\}_{i\in I}$ where $I$ is some indexed with $V_i \in \mathcal{T}$, for all $i \in I$. $V_i \in \mathcal{T}_i$ for all $\mathcal{T}_i$, so $\bigcup_{i\in I} V_i \in \mathcal{T}_i$ for all $\mathcal{T}_i$, and hence must be in $\mathcal{T}$

(ii) For $\Longleftarrow$. Given that $\mathcal{S} \subseteq \langle\mathcal{S}\rangle$, all finite intersections of elements of $\mathcal{S}$ must be in $\mathcal{T}$ by definition. All unions of such sets must also be in $\langle\mathcal{S}\rangle$ by definition.

For $\Longrightarrow$, consider $\beta = \{\text{finite intersections of elements of } \mathcal{S}\}$. We show this is a basis. For all $x \in X$, there exists $B \in \beta$ such that $x \in B$ since $X \in \beta$ (as the intersection of no elements). Given $B_1, B_2 \in \beta$, $B_3 = B_1 \cap B_2 \in \beta$, since the intersection of two finite intersections is itself a finite intersection, so $B_3 \in \beta$. Therefore if $x \in B_1 \cap B_2$, $x \in B_3 \subseteq B_1 \cap B_2$. By Lemma L7-1, $\beta$ is a basis, which generates a topology $\mathcal{T}_\beta$.

By definition $\langle\mathcal{S}\rangle \subseteq \mathcal{T}_\beta$. All elements of $\mathcal{T}_\beta$ are unions of sets in $\beta$ which are in turn finite intersections of elements of $\mathcal{S}$. Therefore for $U$ to be in $\langle\mathcal{S}\rangle$, it must be a union of sets, each of which a finite intersection of elements of $\mathcal{S}$.

(iii) For $\Longrightarrow$ if $f$ is continuous, since $U \in S \in \mathcal{T}$, and is open, so by definition of continuity $f^{-1}(U) \subseteq X$ must be open.

For $\Longleftarrow$, take an arbitrary open set $V \in X$. $V$ is the union of sets, each of which a finite intersection of elements of $\mathcal{S}$.

Consider $f^{-1}(U_1 \cap \cdots \cap U_n) = f^{-1}(U_1) \cap \cdots \cap f^{-1}(U_n)$, $U_i \in \mathcal{S}$. The intersection of a finite number of open sets is open. Also given $A_i$ of the form just described, we have $f^{-1}(A_1 \cup A_2 \cup \dots) = f^{-1}(A_1) \cup f^{-1}(A_2) \cup \dots$, and we just showed that any $f^{-1}(A_i)$ is open, so the union of such sets must also be open.

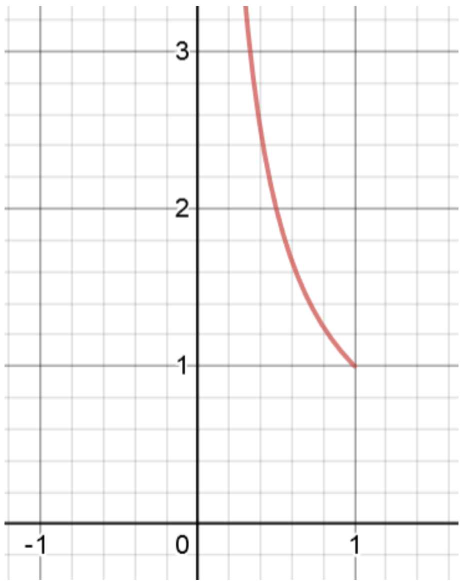
By the previous question, all open sets in $\langle\mathcal{S}\rangle$ are of the form described above, so $f$ must be continuous.

## Exercise L12-12

The map $\mathbb{R}^+ \to (0,1), \quad x \mapsto \tanh(x)$ is a homeomorphism, and composing with $(0,1) \hookrightarrow [0,1] \to [0,1]/\sim = S^1$ embeds $Y = \mathbb{R}$ as a subspace of $S^1$ with complement a point. With $X = (0,1)$ and $f : X \to Y$ given by $f(x) = \frac{1}{x}$ sketch the closed subset of $X \times \tilde{Y} = (0,1) \times S^1$ associated to $f$ by $(\star)$.

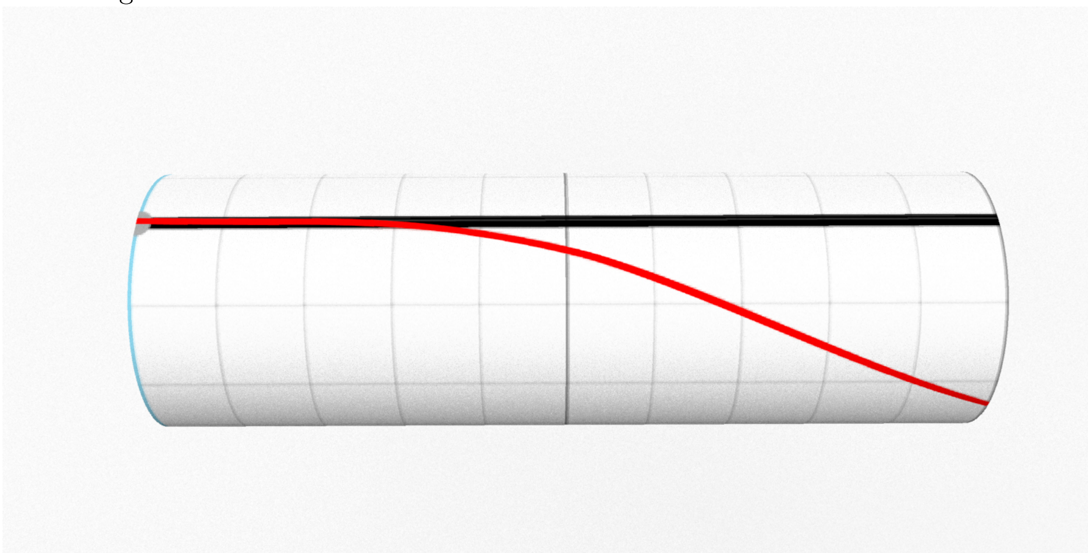
Answer

To start off, we can easily graph $f$, since $Y = \mathbb{R}+$




Now for the transformation from $Y$ to $\tilde{Y}$, we first send $Y \to (0,1)$ by $x \mapsto \tanh(x)$. In other words we have $X \to (0,1)$, $x \mapsto \tanh(\frac{1}{x})$. This is graphed below:

We now just have to send $(0,1) \hookrightarrow [0,1] \to [0,1]/\sim = S^1$. This is defined by the equivalence relation $0 \sim 1$. Graphically representing this, we can take $y = 0$ and $y = 1$ and glue them together as such:



The solid black line is where the gluing has occurred, and going from [0,1] along the 'x' axis. Credit to my housemate Joshua McKinley for using his architecture program to help me render the final cylinder. Note the end points on the graph are open.

## Exercise L13-10

Suppose $X$ is compact and $Y$ is metrisable, with $d_Y^1$, $d_Y^2$ being Lipschitz equivalent metrics inducing the topology. Prove that the two associated metrics $d_\infty^1$, $d_\infty^2$, do on $\text{Cts}(X,Y)$ are also Lipschitz equivalent

Answer

Let $d_Y^1 = d^1$, and $d_Y^2 = d^2$. There exists some $h, k > 0$ such that for any $x, y \in Y$,

$$hd^2(x,y) \leq d^1(x,y) \leq kd^2(x,y)$$

So therefore, for $f, g : X \to Y$,

$$\begin{aligned} hd^2(f(x),g(x)) \leq& d^1(f(x),g(x)) \leq kd^2(f(x),g(x)) \quad \forall x \in X \\ \implies \sup\{hd^2(f(x),g(x))|x \in X\} \leq& \sup\{d^1(f(x),g(x))|x \in X\} \leq \sup\{kd^2(f(x),g(x))|x \in X\} \\ \implies h\sup\{d^2(f(x),g(x))|x \in X\} \leq& \sup\{d^1(f(x),g(x))|x \in X\} \leq k\sup\{d^2(f(x),g(x))|x \in X\} \\ \implies hd_\infty^2(f,g) \leq& d_\infty^1(f,g) \leq kd_\infty^2(f,g) \end{aligned}$$

Therefore $d_\infty^1$, $d_\infty^2$ are Lipschitz equivalent.