Exercise L7-20

In Exercise L10-3 (see below) we prove that $\mathbb{RP}^n \cong S^n/{\sim}$, where $S^n/{\sim}$ is the quotient space on S^n with the equivalence relation defined by $\mathbf{x} \sim \mathbf{y}$ ($\mathbf{x}, \mathbf{y} \in S^n$) if $\mathbf{x} = -\mathbf{y}$ or $\mathbf{x} = \mathbf{y}$. So, to prove that $\mathbb{RP}^n$ is a finite CW-complex, it is equivalent to show that $S^n/{\sim}$ is a CW-complex. We will do this with induction. In fact, we will show that $S^n/{\sim}$ is a CW-complex for all integers $n \geq 0$.

For the base case, notice that $S^0/{\sim}$ must be the one-point space $\{*\}$. This is because the underlying set of S^0 is $\{-1, 1\}$, and $1 \sim -1$ which means $S^0/{\sim}$ has only one element, so must be the one-point space. Letting $X_0 = S^0/{\sim} \cong \{*\}$, clearly X_0 is a finite set with a discrete topology. So $S^0/{\sim}$ is a CW-complex.

Now we show that for each $n \geq 1$, we can obtain $S^n/{\sim}$ from $S^{n-1}/{\sim}$ by attaching an n-cell. Indeed, fix some $n \geq 1$ and let $\rho\colon S^{n-1} \to S^{n-1}/{\sim}$ be the quotient map (note that ρ is continuous since it's a quotient map), where $\sim$ is as defined before. Now consider the following pushout:

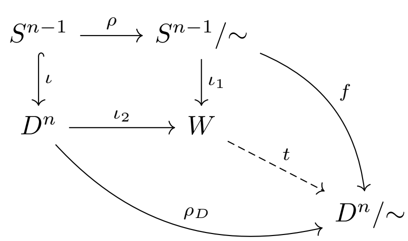
$$\begin{array}{ccc} S^{n-1} & \xrightarrow{\ \rho\ } & S^{n-1}/{\sim} \\ \big\downarrow{\scriptstyle \iota} & & \big\downarrow \\ D^n & \longrightarrow & W \end{array}$$

where $W = S^{n-1}/{\sim} \coprod_{S^{n-1}} D^n$. We'd like to show that $W \cong S^n/{\sim}$.

To see this, we'll first show that $W \cong D^n/{\sim}$, where $D^n/{\sim}$ is the quotient space on D^n with the equivalence relation defined by: for $\mathbf{x}, \mathbf{y} \in D^n$, $\mathbf{x} \sim \mathbf{y}$ if $\mathbf{x}, \mathbf{y} \in S^{n-1}$ and $\mathbf{x} \sim \mathbf{y}$ as elements of S^{n-1}. Note that this definition makes sense (it defines an equivalence relation) since $S^{n-1} \subseteq D^n$. Let $\rho_D\colon D^n \to D^n/{\sim}$ be the corresponding quotient map, which must be continuous. Then, the map $\rho_D \circ \iota\colon S^{n-1} \to D^n/{\sim}$, which is a composition of continuous maps, must also be continuous. Moreover, by our definition of $\sim$ on D^n, we have that whenever $\mathbf{x} \sim \mathbf{y}$ (as elements of S^{n-1}), it is also true that $\iota(\mathbf{x}) \sim \iota(\mathbf{y})$, i.e. $\rho_D(\iota(\mathbf{x})) = \rho_D(\iota(\mathbf{y}))$. So, the universal property of quotient spaces tells us that there exists a unique continuous map $f\colon S^{n-1}/{\sim} \to D^n/{\sim}$ such that $f \circ \rho = \rho_D \circ \iota$. In other words, the following diagram commutes:

$$\begin{array}{ccc} S^{n-1} & \xrightarrow{\ \rho\ } & S^{n-1}/{\sim} \\ \big\downarrow{\scriptstyle \iota} & & \big\downarrow{\scriptstyle f} \\ D^n & \xrightarrow{\ \rho_D\ } & D^n/{\sim} \end{array}$$

So, since f and ρ_D are continuous and the diagram commutes, the universal property of the pushout tells us that there exists a unique continuous map $t\colon W \to D^n/{\sim}$ making the following diagram commute:



i.e. $t \circ \iota_1 = f$ and $t \circ \iota_2 = \rho_D$ where ι_1 and ι_2 are defined in the obvious way. We claim t defines a homeomorphism between W and $D^n/{\sim}$. To prove this, we'll find its inverse. By our definition of $\sim$ on D^n, whenever $\mathbf{x} \sim \mathbf{y}$ (as elements of $D^n/{\sim}$), then $\mathbf{x}, \mathbf{y} \in S^{n-1}$ and $\mathbf{x} \sim \mathbf{y}$ as elements of S^{n-1}. In other words, when $\mathbf{x} \sim \mathbf{y}$ we have

$$\begin{aligned} & \rho(\mathbf{x}) = \rho(\mathbf{y}) && \text{(makes sense since } \mathbf{x}, \mathbf{y} \in S^{n-1}) \\ \implies & \iota_1(\rho(\mathbf{x})) = \iota_1(\rho(\mathbf{y})) && \\ \implies & \iota_2(\iota(\mathbf{x})) = \iota_2(\iota(\mathbf{x})) && \text{(since } \iota_1 \circ \rho = \iota_2 \circ \iota) \\ \implies & \iota_2(\mathbf{x}) = \iota_2(\mathbf{x}) && \end{aligned}$$

where above we abuse notation slightly so that $\mathbf{x}$ may refer to either the element in S^{n-1} or the corresponding element in D^n. Anyway, the point of that is that whenever $\mathbf{x} \sim \mathbf{y}$ (as elements of D^n), we have $\iota_2(\mathbf{x}) = \iota_2(\mathbf{y})$. Since ι_2 is continuous, we may then use the universal property of the quotient space to see that there must be a unique continuous map $g\colon D^n/{\sim} \to W$ such that

$$g \circ \rho_D = \iota_2.$$

We claim that g and t are inverses. Indeed, since $g \circ \rho_D = \iota_2$ and $t \circ \iota_2 = \rho_D$ we have

$$\begin{aligned} & t \circ g \circ \rho_D = t \circ \iota_2 = \rho_D \\ \implies & (t \circ g)(\rho_D(\mathbf{x})) = \rho_D(\mathbf{x}) && \forall \mathbf{x} \in D^n \\ \implies & (t \circ g)(x) = x && \forall x \in D^n/{\sim} \end{aligned}$$

since ρ_D, the quotient map, is surjective. Also since $g \circ \rho_D = \iota_2$ and $t \circ \iota_2 = \rho_D$,

$$\begin{aligned} & g \circ t \circ \iota_2 = g \circ \rho_D = \iota_2 \\ \implies & (g \circ t)(\iota_2(\mathbf{x})) = \iota_2(\mathbf{x}) && \forall \mathbf{x} \in D^n \end{aligned}$$

At this point we just want ι_2 to be surjective. But from the definition of W (as some quotient space of $(S^{n-1}/{\sim}) \coprod D^n$), every element of $w \in W$ can be written in the form $w = \iota_2(\mathbf{x})$ for some $\mathbf{x} \in D^n$ (so that w is in the range of ι_2) or $w = \iota_1(x)$ for some $x \in S^{n-1}/{\sim}$. In the latter case, since $\rho\colon S^{n-1} \to S^{n-1}/{\sim}$ is the quotient map and hence surjective, there must be some $\mathbf{x} \in S^{n-1}$ such that $x = \rho(\mathbf{x})$. Then, using $\iota_1 \circ \rho = \iota_2 \circ \iota$, we get $w = \iota_1(\rho(\mathbf{x})) = \iota_2(\iota(\mathbf{x}))$, which is still in the range of ι_2. Thus, every $w \in W$ is in the range of ι_2, so ι_2 is surjective. Then

$$(g \circ t)(\iota_2(\mathbf{x})) = \iota_2(\mathbf{x}) \ \forall \mathbf{x} \in D^n \implies (g \circ t)(x) = \iota_2(x) \ \forall x \in W.$$

So, g and t are inverses, and since both are continuous we can conclude that t is a homeomorphism. So we have $W \cong D^n/{\sim}$.

All that remains is to show that $D^n/{\sim} \cong S^n/{\sim}$. To see that this is true, consider the following maps:

$$\begin{aligned} & h\colon D^n \to S^n && \text{given by } h((x_1, x_2, \ldots, x_n)) = (x_1, x_2, \ldots, x_n, \sqrt{1 - x_1^2 - x_2^2 - \cdots - x_n^2}) \\ & \bar{h}\colon D^n \to S^n/{\sim} && \text{given by } \bar{h}((x_1, x_2, \ldots, x_n)) = \rho_S(h((x_1, x_2, \ldots, x_n))) \end{aligned}$$

where $\rho_S\colon S^n \to S^n/{\sim}$ is the quotient map (which is continuous). h is well-defined, since if $(x_1, x_2, \ldots, x_n) \in D^n$ then $x_1^2 + x_2^2 + \cdots + x_n^2 = \|(x_1, x_2, \ldots, x_n)\|^2 \leq 1$ so $\sqrt{1 - x_1^2 - x_2^2 - \cdots - x_n^2}$ is indeed a real number, and it is easily verified that $\|(x_1, x_2, \ldots, x_n, \sqrt{1 - x_1^2 - x_2^2 - \cdots - x_n^2})\| = 1$. Also, h is continuous: to see this, first consider the corresponding function $h_2\colon D^n \to \mathbb{R}^{n+1}$ defined by $h_2(\mathbf{x}) = h(x)$. Each component of h_2 is continuous as they only involve squares and square roots, so then by the universal property of product spaces (on $\mathbb{R}^{n+1}$), h_2 is continuous. It is clear then that h must also be continuous. Then, $\bar{h} = \rho_S \circ h$ is a composition of continuous functions so is also continuous. Also, for all $\mathbf{x}, \mathbf{y} \in D^n$ we have

$$\begin{aligned} \mathbf{x} \sim \mathbf{y} &\implies \mathbf{x}, \mathbf{y} \in S^{n-1} \text{ and } \mathbf{x} = \pm\mathbf{y} \\ &\implies h(\mathbf{x}) = (\mathbf{x}, 0) = \pm(\mathbf{y}, 0) = \pm h(\mathbf{y}) \\ &\implies \bar{h}(\mathbf{x}) = \bar{h}(\mathbf{y}) \end{aligned}$$

where above we abuse notation slightly so as to keep things neater to read: by $(\mathbf{x}, 0)$ we mean $(x_1, x_2, \ldots, x_n, 0)$, where $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, and note also that $h(\mathbf{x}) = (\mathbf{x}, 0)$ in the second line is a consequence of $\mathbf{x} \in S^{n-1}$ (which implies that $\|\mathbf{x}\| = 1$). Anyway, the point is that whenever $\mathbf{x} \sim \mathbf{y}$ (as elements of D^n) we also have $\bar{h}(\mathbf{x}) = \bar{h}(\mathbf{y})$, so since $\bar{h}$ is continuous, by the universal property of the quotient space there must be a unique continuous map $H\colon D^n/{\sim} \to S^n/{\sim}$ such that

$$\rho_S \circ h = \bar{h} = H \circ \rho_D.$$

We'll now prove that H is actually a homeomorphism between $D^n/\sim$ and $S^n/\sim$.

Firstly, H surjective since we can write any element in $S^n/\sim$ as $\rho_S((x_1, x_2, \dots, x_{n+1})$ for some $(x_1, x_2, \dots, x_{n+1}) \in S^n$, such that $x_{n+1} \geq 0$ (we can assume this by the nature of $\sim$ since if $x_{n+1} < 0$ then we can multiply the entire element by -1 and leave the equivalence class unchanged), and then we have

$$H(\rho_D((x_1, x_2, \dots, x_n))) = \rho_S((x_1, x_2, \dots, x_n, \sqrt{1 - x_1^2 - x_2^2 - \dots - x_n^2})) = \rho_S((x_1, x_2, \dots, x_n, x_{n+1}))$$

since $\|(x_1, x_2, \dots, x_n, x_{n+1})\| = 1$ and $x_{n+1} \geq 0$.

Secondly, H is injective since for all $\mathbf{x}, \mathbf{y} \in D^n$, if $H(\rho_D(\mathbf{x})) = H(\rho_D(\mathbf{y}))$ then

$$\rho_S((\mathbf{x}, \sqrt{1 - \|\mathbf{x}\|^2})) = \rho_S((\mathbf{y}, \sqrt{1 - \|\mathbf{y}\|^2}))$$

(slight abuse of notation again, but it should be clear what it means) then we have one of the following scenarios:

- if $\sqrt{1 - \|\mathbf{x}\|^2} = 0$ we must also have $\sqrt{1 - \|\mathbf{y}\|^2} = 0$ and then $\mathbf{x} = \pm\mathbf{y}$ by the definition of $\sim$ for S^n. This all implies $\|\mathbf{x}\|, \|\mathbf{y}\| = 1$ and so $\mathbf{x}, \mathbf{y} \in S^{n-1}$. All of this together implies $\rho_D(\mathbf{x}) = \rho_D(\mathbf{y})$.
- otherwise $\sqrt{1 - \|\mathbf{x}\|^2} > 0$, which would then imply $\sqrt{1 - \|\mathbf{y}\|^2} = \sqrt{1 - \|\mathbf{x}\|^2} > 0$ from which we also conclude $\mathbf{x} = \mathbf{y}$, so $\rho_D(\mathbf{x}) = \rho_D(\mathbf{y})$ is also true.

So, whenever $H(x) = H(y)$ for some $x, y \in D^n/\sim$ we must also have $x = y$. Hence H is injective.

So now we know that H is a continuous bijection. Also, $D^n/\sim$ is compact by Corollary L10-4, and we can also show $S^n/\sim$ is Hausdorff: for any $x, y \in S^n/\sim$ such that $x \neq y$, we can pick $\mathbf{x}, \mathbf{y} \in S^n$ such that $\mathbf{x} \neq \mathbf{y}$, $\mathbf{x} \neq -\mathbf{y}$ and $x = \rho_S(\mathbf{x})$ and $y = \rho_S(\mathbf{y})$. Now, letting $\epsilon = \frac{1}{2}\min\{\|\mathbf{x} - \mathbf{y}\|, \|\mathbf{x} + \mathbf{y}\|, 1\}$, which is strictly positive since $\mathbf{x} \neq \mathbf{y}$ and $\mathbf{x} \neq -\mathbf{y}$, we'll define the following open sets in S^n:

$$U_1 = B_\epsilon(\mathbf{x}) \cap S^n, \qquad V_1 = B_\epsilon(\mathbf{y}) \cap S^n, \qquad U_2 = B_\epsilon(-\mathbf{x}) \cap S^n, \qquad V_2 = B_\epsilon(-\mathbf{y}) \cap S^n$$

(where $B_\epsilon(\mathbf{x}), B_\epsilon(\mathbf{y}), B_\epsilon(-\mathbf{x}), B_\epsilon(-\mathbf{y})$ are open balls in $\mathbb{R}^{n+1}$, so the above are indeed open in the subspace topology S^n). From our choice of ϵ it is clear that the four sets are pairwise disjoint, so also $U_1 \cup U_2$ and $V_1 \cup V_2$ are disjoint. Now let $U = \rho_S(U_1)$ and $V = \rho_S(V_1)$, then clearly

$$\rho_S^{-1}(U) = U_1 \cup U_2, \qquad \rho_S^{-1}(V) = V_1 \cup V_2$$

and then we see that U, V must be open by the definition of the quotient topology (since $U_1 \cup U_2$ and $V_1 \cup V_2$ are unions of open sets so are open), and moreover must be disjoint as their preimages are disjoint. It is also clear that $x = \rho_S(\mathbf{x}) \in U$ and $y = \rho_S(\mathbf{y}) \in V$, thus $S^n/\sim$ is Hausdorff as claimed.

Anyway, the point is that $H\colon D^n/\sim \to S^n/\sim$ is a continuous bijection from a compact space to a Hausdorff space, so by Lemma L11-6 H is a homeomorphism and so $D^n/\sim \cong S^n/\sim$.

We are now done, but just to summarise everything: we proved $S^0/\sim$ is a CW-complex, then we showed that $D^n/\sim$ can be obtained from $S^{n-1}/\sim$ by attaching a single n-cell. Then, we showed $S^n/\sim \cong D^n/\sim$, so actually it is possible to obtain $S^n/\sim$ from $S^{n-1}/\sim$ by attaching a single n-cell. By induction, it then follows that each $S^n/\sim$, $n \geq 0$, is a finite CW-complex. Finally, since $\mathbb{RP}^n \cong S^n/\sim$ (as proved in the next question), $\mathbb{RP}^n$ must also be a finite CW-complex.

Exercise L10-3

Recall that for $n \geq 1$, the real projective space is defined to be the quotient space $\mathbb{RP}^n = (\mathbb{R}^{n+1}\backslash\{\mathbf{0}\})/\sim$ where $(a_0, a_1, \dots, a_n) \sim (b_0, b_1, \dots, b_n)$ if there exists $\lambda \in \mathbb{R}\backslash\{0\}$ with $\lambda a_i = b_i$ for all $0 \leq i \leq n$. Call the corresponding quotient map $\rho_1\colon \mathbb{R}^{n+1}\backslash\{\mathbf{0}\} \to \mathbb{RP}^n$.

Consider the quotient space $S^n/\sim$ where $\mathbf{x} \sim \mathbf{y}$ ($\mathbf{x}, \mathbf{y} \in S^n$) if $\mathbf{x} = -\mathbf{y}$ or $\mathbf{x} = \mathbf{y}$ (clearly $\sim$ is reflexive, symmetric and transitive). Call the corresponding quotient map $\rho_2\colon S^n \to S^n/\sim$.

We claim that $S^n/{\sim} \cong \mathbb{RP}^n$. Indeed, consider the following functions:

$$\begin{aligned} &f\colon S^n \to \mathbb{R}^{n+1}\backslash\{\mathbf{0}\} && \text{given by } f(\mathbf{x}) = \mathbf{x} \\ &\bar{f}\colon S^n \to \mathbb{RP}^n && \text{given by } \bar{f}(\mathbf{x}) = \rho_1(f(\mathbf{x})) \end{aligned}$$

i.e. f is the inclusion $S^n \to \mathbb{R}^{n+1}\backslash\{\mathbf{0}\}$, and $\bar{f}$ is the composition $\rho_1 \circ f$. Note that f is well-defined since $S^n \subseteq \mathbb{R}^{n+1}$ and $\|\mathbf{x}\| = 1 \neq 0$ for all $\mathbf{x} \in S^n$ so we do have $S^n \subseteq \mathbb{R}^{n+1}\backslash\{\mathbf{0}\}$. Now, since f is an inclusion map it must be continuous (by the definition of subspace topology). Then, since ρ_1 is also continuous (by the definition of quotient space), $\bar{f} = \rho_1 \circ f$ must also be continuous as it is the composition of continuous functions. Moreover, for all $\mathbf{x}, \mathbf{y} \in S^n$ we have

$$\mathbf{x} \sim \mathbf{y} \implies \mathbf{x} = \mathbf{y} \text{ or } \mathbf{x} = -\mathbf{y} \implies \exists \lambda \in \mathbb{R} \text{ s.t. } \mathbf{x} = \lambda \mathbf{y}$$

(i.e. either $\lambda = 1$ or $\lambda = -1$ will do). Then if $\mathbf{x} \sim \mathbf{y}$ (as elements in S^n), there exists λ such that $f(\mathbf{x}) = \mathbf{x} = \lambda\mathbf{y} = \lambda f(\mathbf{y})$, which implies $f(\mathbf{x}) \sim f(\mathbf{y})$ (as elements in $\mathbb{R}^{n+1}$), so that $\bar{f}(\mathbf{x}) = \rho_1(f(\mathbf{x})) = \rho_1(f(\mathbf{y})) = \bar{f}(\mathbf{y})$ whenever $\mathbf{x} \sim \mathbf{y}$. Then (recalling that $\bar{f}$ is continuous), from the universal property of quotient spaces, there must exist a unique continuous map $F\colon S/{\sim} \to \mathbb{RP}^n$ such that

$$\bar{f} = F \circ \rho_2.$$

We claim that F is a homeomorphism $S/{\sim} \to \mathbb{RP}^n$. We already know F is continuous, so now it suffices to show F has a continuous inverse. To go about constructing this inverse, we'll consider the following functions:

$$\begin{aligned} &g\colon \mathbb{R}^{n+1}\backslash\{\mathbf{0}\} \to S^n && \text{given by } g(\mathbf{x}) = \frac{\mathbf{x}}{\|\mathbf{x}\|} \\ &\bar{g}\colon \mathbb{R}^{n+1}\backslash\{\mathbf{0}\} \to S^n/{\sim} && \text{given by } \bar{g}(\mathbf{x}) = \rho_2(g(\mathbf{x})). \end{aligned}$$

Clearly g is well-defined since for all $\mathbf{x} \in \mathbb{R}^{n+1}\backslash\{\mathbf{0}\}$ we have $\left\|\frac{\mathbf{x}}{\|\mathbf{x}\|}\right\| = \frac{\|\mathbf{x}\|}{\|\mathbf{x}\|} = 1$. Moreover, since $\|-\|$ is continuous and non-zero in $\mathbb{R}^{n+1}\backslash\{\mathbf{0}\}$, we see that g must also be continuous. Then, since ρ_2 is continuous (by the definition of quotient space), $\bar{g} = \rho_2 \circ g$ must also be continuous as it is the composition of continuous functions. Also, for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{n+1}\backslash\{\mathbf{0}\}$ we have

$$\begin{aligned} \mathbf{x} \sim \mathbf{y} &\implies \exists \lambda \in \mathbb{R} \text{ s.t. } \mathbf{x} = \lambda \mathbf{y} \\ &\implies g(\mathbf{x}) = \frac{\mathbf{x}}{\|\mathbf{x}\|} = \frac{\lambda\mathbf{y}}{\|\lambda\mathbf{y}\|} \end{aligned}$$

and either $\frac{\lambda\mathbf{y}}{\|\lambda\mathbf{y}\|} = \frac{\mathbf{y}}{\|\mathbf{y}\|} = g(\mathbf{y})$ or $\frac{\lambda\mathbf{y}}{\|\lambda\mathbf{y}\|} = -\frac{\mathbf{y}}{\|\mathbf{y}\|} = -g(\mathbf{y})$. So if $\mathbf{x} \sim \mathbf{y}$ (as elements in $\mathbb{R}^{n+1}\backslash\{\mathbf{0}\}$) we have $g(\mathbf{x}) = g(\mathbf{y})$ or $g(\mathbf{x}) = -g(\mathbf{y})$. Either way, we will have $g(\mathbf{x}) \sim g(\mathbf{y})$ (as elements in S^n), so that

$$\bar{g}(\mathbf{x}) = \rho_2(g(\mathbf{x})) = \rho_2(g(\mathbf{y})) = \bar{g}(\mathbf{y})$$

whenever $\mathbf{x} \sim \mathbf{y}$. Thus (recalling that $\bar{g}$ is continuous), from the universal property of quotient spaces there must exist a unique continuous map $G\colon (\mathbb{R}^{n+1}\backslash\{\mathbf{0}\})/{\sim} \to S^n/{\sim}$ such that

$$\bar{g} = G \circ \rho_1.$$

Now we will show that this G is the inverse of F. Indeed, for all $\mathbf{x} \in \mathbb{R}^{n+1}\backslash\{\mathbf{0}\}$ we have

$$\begin{aligned} (F \circ G)(\rho_1(\mathbf{x})) &= (F \circ G \circ \rho_1)(\mathbf{x}) \\ &= (F \circ \rho_2 \circ g)(\mathbf{x}) && \text{(since } G \circ \rho_1 = \bar{g} = \rho_2 \circ g) \\ &= (\rho_1 \circ f \circ g)(\mathbf{x}) && \text{(since } F \circ \rho_2 = \bar{f} = \rho_1 \circ f) \\ &= \rho_1(f(g(\mathbf{x}))) \\ &= \rho_1\left(\frac{\mathbf{x}}{\|\mathbf{x}\|}\right) && \text{(using the definition of } f \text{ and } g) \\ &= \rho_1(\mathbf{x}) \end{aligned}$$

since it is clear that $\mathbf{x} \sim \frac{\mathbf{x}}{\|\mathbf{x}\|}$ (as elements of $\mathbf{x} \in \mathbb{R}^{n+1}\backslash\{\mathbf{0}\}$, you'd just take $\lambda = \frac{1}{\|\mathbf{x}\|}$ in the given definition of $\sim$), so since ρ_1 is the quotient map and hence surjective, we have that $(F \circ G)(x) = x$ for all $x \in (\mathbb{R}^{n+1}\backslash\{\mathbf{0}\})/\sim$. Similarly, for all $\mathbf{y} \in S^n$ we have

$$\begin{aligned}
(G \circ F)(\rho_2(\mathbf{y})) &= (G \circ F \circ \rho_2)(\mathbf{y}) \\
&= (G \circ \rho_1 \circ f)(\mathbf{y}) && \text{(since } F \circ \rho_2 = \bar{f} = \rho_1 \circ f\text{)} \\
&= (\rho_2 \circ g \circ f)(\mathbf{y}) && \text{(since } G \circ \rho_1 = \bar{g} = \rho_2 \circ g\text{)} \\
&= \rho_2(g(f(\mathbf{y}))) \\
&= \rho_2\left(\frac{\mathbf{y}}{\|\mathbf{y}\|}\right) && \text{(using the definition of } f \text{ and } g\text{)} \\
&= \rho_2(\mathbf{y}) && \text{(since } \mathbf{y} \in S^n \text{ implies } \|\mathbf{y}\| = 1\text{).}
\end{aligned}$$

Then, since ρ_2 is the quotient map and hence surjective, the above actually shows that $(G \circ F)(y) = y$ for all $y \in S^n/\sim$.

Therefore, F and G are indeed inverses, and since both are continuous we now see that $S^n/\sim \,\cong (\mathbb{R}^{n+1}\backslash\{\mathbf{0}\})/\sim = \mathbb{RP}^n$ as we had initially claimed.

Now, Corollary L10-4 tells us S^n is compact, then Lemma L10-1 tells us $S^n/\sim$ must also be compact. Finally, since $S^n/\sim$ and $\mathbb{RP}^n$ are homeomorphic (as we just proved), Exercise L10-1 tells us that since $S^n/\sim$ is compact, $\mathbb{RP}^n$ must also be compact. Or, if I can't quote the exercise, we can instead use Proposition L9-3 on F to conclude that $F(S^n/\sim) = \mathbb{RP}^n$ is compact. (Note that the image of F is indeed $\mathbb{RP}^n$ since F has an inverse, G, so F is a bijection, which must be surjective).

Exercise L12-2

Let $f\colon X_1 \to Y_1$ and $g\colon X_2 \to Y_2$ be continuous maps. We would like to show that the map

$$f \times g\colon X_1 \times X_2 \to Y_1 \times Y_2, \text{ given by } (x_1, x_2) \mapsto (f(x_1), g(x_2))$$

is continuous.

From the definition of product spaces, the set $\mathcal{B} = \{U \times V \mid U \subseteq Y_1 \text{ open in } Y_1,\ V \subseteq Y_2 \text{ open in } Y_2\}$ is a basis for the topology associated with $Y_1 \times Y_2$. So, from Exercise L7-1 part (ii) (see Assignment 1), to show that $f \times g$ is continuous, it suffices to show that $(f \times g)^{-1}(B)$ is open in $X_1 \times X_2$ for all $B \in \mathcal{B}$.

Let $B \in \mathcal{B}$, then B can be written as $U \times V$ for some $U \subseteq Y_1$, $V \subseteq Y_2$ open. Then

$$\begin{aligned}
(f \times g)^{-1}(B) &= (f \times g)^{-1}(U \times V) \\
&= \{(x_1, x_2) \in X_1 \times X_2 \mid (f \times g)(x_1, x_2) \in U \times V\} \\
&= \{(x_1, x_2) \in X_1 \times X_2 \mid (f(x_1), g(x_2)) \in U \times V\} \\
&= \{(x_1, x_2) \in X_1 \times X_2 \mid f(x_1) \in U,\ g(x_2) \in V\} \\
&= \{(x_1, x_2) \in X_1 \times X_2 \mid x_1 \in f^{-1}(U),\ x_2 \in g^{-1}(V)\} \\
&= f^{-1}(U) \times g^{-1}(V).
\end{aligned}$$

Now, since $U \subseteq Y_1$ is open and $f\colon X_1 \to Y_1$ is continuous, $f^{-1}(U)$ must be open in X_1. Similarly, since $V \subseteq Y_2$ is open and $g\colon X_2 \to Y_2$ is continuous, $g^{-1}(V)$ must be open in X_2. Then, by the definition of the topology of a product space, $f^{-1}(U) \times g^{-1}(V)$ must be open in $X_1 \times X_2$. Thus we have shown that $(f \times g)^{-1}(B) = f^{-1}(U) \times g^{-1}(V)$ is open, and since this holds for any $B \in \mathcal{B}$, $f \times g$ must be continuous (as explained earlier).

Exercise L12-5

Call the map given in the question Ψ, i.e. let $\Psi\colon \operatorname{Cts}(X, Y) \to \prod_{x \in X} Y$ be the map given by

$$\Psi(f) = (f(x))_{x \in X}.$$

We want to prove that Ψ is a homeomorphism. Let's first prove that it is a bijection:

Ψ is injective: Suppose $f, g \in \text{Cts}(X, Y)$ are such that $\Psi(f) = \Psi(g)$. We have:

$$\Psi(f) = \Psi(g) \implies (f(x))_{x\in X} = (g(x))_{x\in X} \implies f(x) = g(x)\ \forall x \in X \implies f = g$$

Hence Ψ is injective.

Ψ is surjective: Consider any $(a_x)_{x\in X} \in \prod_{x\in X}$. Consider the function $f\colon X \to Y$ defined by

$$f(x) = a_x\ \forall x \in X.$$

Since X is discrete (i.e. every subset of X is open), it immediately follows that $f^{-1}(A)$ is open for every open set $A \subseteq \prod_{x\in X} Y$. Hence f is continuous, so since $f \in \text{Cts}(X, Y)$ and $\Psi(f) = (f(x))_{x\in X} = (a_x)_{x\in X}$, we see that Ψ is surjective.

So, since Ψ is injective and surjective, it must be a bijection. We now need to prove that Ψ and Ψ^{-1} are both continuous. To do this, we'll make use of the following result:

"Ψ identifies $S(x, U)$ with $\pi_x^{-1}(U)$": More precisely, we will show that for any open subset $U \subseteq Y$ and $x \in X$, that $\Psi(S(x, U)) = \pi_x^{-1}(U)$ and $\Psi^{-1}(\pi_x^{-1}(U)) = S(x, U)$ (where $\pi_x\colon \prod_{x\in X} Y \to Y$ is the projection and by $S(x, U)$ we mean $S(\{x\}, U)$). So, given any $x \in X$ and open $U \subseteq Y$, we have

$$\begin{aligned}
\Psi(S(x, U)) &= \Psi(\{f \in \text{Cts}(X, Y) \mid f(x) \in U\}) \\
&= \{(a_z)_{z\in X} \in \prod_{z\in X} Y \mid a_x \in U\} \\
&= \{(a_z)_{z\in X} \in \prod_{z\in X} Y \mid \pi_x((a_z)_{z\in X}) \in U\} \\
&= \pi_x^{-1}(U)
\end{aligned}$$

and

$$\begin{aligned}
\Psi^{-1}(\pi_x^{-1}(U)) &= \{f \in \text{Cts}(X, Y) \mid (f(z))_{z\in X} \in \pi_x^{-1}(U)\} \\
&= \{f \in \text{Cts}(X, Y) \mid \pi_x((f(z))_{z\in X}) \in U\} \\
&= \{f \in \text{Cts}(X, Y) \mid f(x) \in U\} \\
&= S(x, U)
\end{aligned}$$

as required.

Now we'll get onto proving Ψ and Ψ^{-1} are continuous.

Ψ is continuous: By the definition of the product space, a basis $\mathcal{B}$ for $\prod_{x\in X} Y$ is given by sets of the form

$$\prod_{x\in X} U_x$$

where each $U_x \subseteq Y$ is open. (Note that usually there is the condition that $U_x \neq Y$ for finitely many $x \in X$, but since X is finite this is true for any of the products of the above form). Now, from Exercise L7-1 (ii) (see proof in Assignment 1), to prove that f is continuous it suffices to prove that $\Psi^{-1}(B)$ is open for each $B \in \mathcal{B}$. Each set $B \in \mathcal{B}$ can be written in the form $\prod_{x\in X} U_x$ (where $U_x \subseteq Y$ is open for each $x \in X$), so we have

$$\begin{aligned}
\Psi^{-1}\left(\prod_{x\in X} U_x\right) &= \{f \in \mathrm{Cts}(X,Y) \mid (f(x))_{x\in X} \in \prod_{x\in X} U_x\} \\
&= \{f \in \mathrm{Cts}(X,Y) \mid f(x) \in U_x \ \forall x \in X\} \\
&= \bigcap_{x\in X} \{f \in \mathrm{Cts}(X,Y) \mid f(x) \in U_x\} \\
&= \bigcap_{x\in X} \{f \in \mathrm{Cts}(X,Y) \mid (f(z))_{z\in X} \in \pi_x^{-1}(U_x)\} \\
&= \bigcap_{x\in X} \Psi^{-1}\left(\pi_x^{-1}(U_x)\right).
\end{aligned}$$

But from what we previously proved, we know that $\Psi^{-1}\left(\pi_x^{-1}(U_x)\right) = S(x, U_x)$ which is open (by the definition of the compact-open space, since $\{x\}$ is one element and hence compact, while U_x was said to be open earlier), so since $\Psi^{-1}\left(\prod_{x\in X} U_x\right)$ is a finite intersection (since X is finite) of these open sets, it follows that $\Psi^{-1}\left(\prod_{x\in X} U_x\right)$ is open. Thus Ψ is continuous.

Ψ^{-1} **is continuous:** To prove this, we'll make use of part of the result from Exercise L12-3 (iii), i.e. if $f\colon A \to B$ is a function and S is a sub-basis for the topology on B, then f is continuous if $f^{-1}(U)$ is open for every $U \in S$. Now, from the definition of the compact-open subspace, the set $\{S(K,U) \mid K \subseteq X \text{ compact}, U \subseteq Y \text{ open}\}$ is a sub-basis for $\mathrm{Cts}(X,Y)$, so since Ψ is the inverse of Ψ^{-1} we just need to show that $\Psi(S(K,U))$ is open whenever $K \subseteq X$ is compact and $U \subseteq Y$ is open. But, since X is finite, we can write $K = \{x_1, x_2, \ldots, x_n\}$ so that

$$\begin{aligned}
\Psi(S(K,U)) &= \Psi(\{f \in \mathrm{Cts}(X,Y) \mid f(x) \in U \ \forall x \in K\}) \\
&= \{(a_x)_{x\in X} \in \prod_{x\in X} Y \mid a_x \in U \ \forall x \in K\} \\
&= \{(a_x)_{x\in X} \in \prod_{x\in X} Y \mid a_{x_i} \in U, 1 \le i \le n\} \\
&= \bigcap_{1\le i\le n} \{(a_x)_{x\in X} \in \prod_{x\in X} Y \mid a_{x_i} \in U\} \\
&= \bigcap_{1\le i\le n} \Psi(\{f \in \mathrm{Cts}(X,Y) \mid f(x_i) \in U\}) \\
&= \bigcap_{1\le i\le n} \Psi(S(x_i, U))
\end{aligned}$$

But from what we've previously proved, we know that that $\Psi(S(x_i,U)) = \pi_{x_i}^{-1}(U)$, which is open as it is the preimage of an open set U under a continuous function π_x. So, since $\Psi(S(K,U))$ is the finite intersection of sets of this form, it follows that $\Psi(S(K,U))$ is open. Thus Ψ^{-1} is continuous as we discussed.

To conclude, Ψ is a continuous bijection with a continuous inverse, hence it is a homeomorphism.

Lemma for the next two questions

The next two questions both involve proving that some function $\Psi\colon A \to B$, where A and B are topological spaces, is a homeomorphism onto its image. Suppose Q is a sub-basis for the topology on A. Here we will show that proving the following is enough to to prove Ψ is a homeomorphism onto its image:

- Ψ is continuous, and

- Ψ is injective, and
- For all open sets U in Q, $\Psi(U)$ is open in $\operatorname{Im}\Psi$ (which is given the subspace topology, $\operatorname{Im}\Psi \subseteq B$).

Indeed, suppose all three of the above conditions are satisfied. We want to show that the function $\Psi'\colon A \to \operatorname{Im}\Psi$ given by $\Psi'(x) = \Psi(x)$ for all $x \in A$ is a homeomorphism. Now, by the definition of $\operatorname{Im}\Psi$ and Ψ', it is clear that Ψ' must be surjective. Moreover, since Ψ is injective, we have, for $x, y \in A$,

$$\Psi'(x) = \Psi'(y) \implies \Psi(x) = \Psi(y) \implies x = y$$

so Ψ' is injective. Hence Ψ' is a bijection. It remains to show that both Ψ' and its inverse are continuous.

To show that Ψ' is continuous, let U be any open set of $\operatorname{Im}\Psi$. Then $U = V \cap \operatorname{Im}\Psi$ for some open $V \subseteq B$, by the definition of the subspace topology. Then

$$\Psi'^{-1}(U) = \Psi^{-1}(U) = \Psi^{-1}(V \cap \operatorname{Im}\Psi) = \Psi^{-1}(V) \cap \Psi^{-1}(\operatorname{Im}\Psi) = \Psi^{-1}(V) \cap A = \Psi^{-1}(V)$$

which must be open in A since Ψ is continuous and V is open in B. Hence Ψ' is continuous.

Finally, to show that the inverse of Ψ' is continuous, it suffices to show that the images under Ψ' of all open sets in some sub-basis of A are open in $\operatorname{Im}\Psi$. But this is immediate from the third dot point above (noting that $\Psi'(U) = \Psi(U)$ for all subsets $U \subseteq A$). Hence the inverse of Ψ' is continuous.

Hence, Ψ' is a continuous bijection with a continuous inverse, so is a homeomorphism. So, Ψ is a homeomorphism onto its image, as we wanted to show.

Exercise L12-11

Let X, Y_1, Y_2 and ι be as given in the problem statement. Let $\Psi\colon \operatorname{Cts}(X, Y_1) \to \operatorname{Cts}(X, Y_2)$ be the map given by

$$\Psi(f) = \iota \circ f.$$

Then, we are required to prove that Ψ is a homeomorphism onto its image. By the lemma in the previous section, it suffices to show that Ψ is continuous, injective, and sends all open sets in some sub-basis of $\operatorname{Cts}(X, Y_1)$ to open sets in the image of Ψ.

First, notice that since $\iota\colon Y_1 \to Y_2$ is continuous and X is locally compact Hausdorff, by Lemma 12.1 (iii) Ψ must be continuous. Also, Ψ is injective since if $\Psi(f) = \Psi(g)$ for some $f, g \in \operatorname{Cts}(X, Y_1)$ then

$$\Psi(f) = \Psi(g) \implies \iota \circ f = \iota \circ f \implies \iota(f(x)) = \iota(g(x))\ \forall x \in X \implies f(x) = g(x)\ \forall x \in X \implies f = g$$

where the second last implication follows from ι being injective (ι is injective since it is an inclusion map).

Finally, note that the set $\{S(K, U) \mid K \subseteq X \text{ compact}, U \subseteq Y_1 \text{ open}\}$ is a sub-basis for the topology on $\operatorname{Cts}(X, Y_1)$, so we just need to show that $\Psi(S(K, U))$ is open whenever $K \subseteq X$ is compact and $U \subseteq Y_1$ is open. So, let $K \subseteq X$ be compact and $U \subseteq Y_1$ be open, then $U = V \cap Y_1$ for some open set $V \subseteq Y_2$. We have

$$\begin{aligned}
&\Psi(S(K, U)) \\
&= \Psi(\{f \in \operatorname{Cts}(X, Y_1) \mid f(K) \subseteq U\}) \\
&= \{g \in \operatorname{Cts}(X, Y_2) \mid g(K) \subseteq U \text{ and } g \in \operatorname{Im}\Psi\} \\
&= \{g \in \operatorname{Cts}(X, Y_2) \mid g(K) \subseteq V \cap Y_1 \text{ and } g(X) \subseteq Y_1\} && (\text{since } U = V \cap Y_1) \\
&= \{g \in \operatorname{Cts}(X, Y_2) \mid g(K) \subseteq V \text{ and } g(K) \subseteq Y_1 \text{ and } g(X) \subseteq Y_1\} \\
&= \{g \in \operatorname{Cts}(X, Y_2) \mid g(K) \subseteq V \text{ and } g(X) \subseteq Y_1\} && (g(K) \subseteq Y_1 \text{ redundant since } g(X) \subseteq Y_1) \\
&= \{g \in \operatorname{Cts}(X, Y_2) \mid g(K) \subseteq V\} \cap \{g \in \operatorname{Cts}(X, Y_2) \mid g(X) \subseteq Y_1\} \\
&= S(K, V) \cap \operatorname{Im}\Psi
\end{aligned}$$

and since $K \subseteq Y_1 \subseteq Y2$ is compact and $V \subseteq Y_2$ is open, $S(K, V)$ is open by the definition of compact-open topology on $\mathrm{Cts}(X, Y_2)$. So, $S(K, V) \cap \operatorname{Im} \Psi$ is open in the subspace topology on $\operatorname{Im} \Psi$, which means that $\Psi(S(K, U))$ is open.

So, Ψ satisfies all three dot points of our lemma from the previous section, so Ψ is a homeomorphism onto its image.

Exercise L13-2

(i)

Let Y, X, $\sim$ and ρ be as given in the question. Let $\Psi\colon \mathrm{Cts}(X/{\sim}, Y) \to \mathrm{Cts}(X, Y)$ be the map given by

$$\Psi(f) = f \circ \rho.$$

Then we are required to prove that Ψ is a homeomorphism onto its image. By the lemma we proved before the previous question, it suffices to show that that Ψ is continuous, injective and maps all open sets in some sub-basis of $\mathrm{Cts}(X/{\sim}, Y)$ to open sets of $\operatorname{Im} \Psi$ (with the subspace topology).

So now, the question statement already mentions why Ψ is continuous. And to see that Ψ is injective, suppose $f, g \in \mathrm{Cts}(X/{\sim}, Y)$ are such that $\Psi(f) = \Psi(g)$. Then

$$\Psi(f) = \Psi(g) \implies f \circ \rho = g \circ \rho \implies f(\rho(x)) = g(\rho(x)) \ \forall x \in X \implies f = g$$

where above we are using the fact that ρ is surjective (it is a quotient map), so that any element of $X/{\sim}$ can be written in the form $\rho(x)$ for some $x \in X$. Anyway, this proves that Ψ is injective.

Now, $\{S(K, U) \mid K \subseteq X/{\sim} \text{ compact}, U \subseteq Y \text{ open}\}$ a sub-basis for $\mathrm{Cts}(X/{\sim}, Y)$ by the definition of compact-open topology, so we just need to show that $\Psi(S(K, U))$ is open in $\Psi(X/{\sim})$ whenever $K \subseteq X/{\sim}$ is compact and $U \subseteq Y$ is open. We have

$$\begin{aligned}
\Psi(S(K, U)) &= \Psi(\{f \in \mathrm{Cts}(X/{\sim}, Y) \mid f([x]) \in U \text{ whenever } [x] \in K\}) \\
&= \{g \in \mathrm{Cts}(X, Y) \mid (g(x) \in U \text{ whenever } \rho(x) \in K) \text{ and } g \in \operatorname{Im} \Psi\} \\
&= \{g \in \mathrm{Cts}(X, Y) \mid g(x) \in U \text{ whenever } x \in \rho^{-1}(K)\} \cap \operatorname{Im} \Psi \\
&= S(\rho^{-1}(K), U) \cap \operatorname{Im} \Psi
\end{aligned}$$

Now, since $X/{\sim}$ was said to be Hausdorff and $K \subseteq X/{\sim}$ is compact, then by Lemma L11-5 we have that K must be closed. Then,

$$\begin{aligned}
\rho^{-1}(K) &= \{x \in X \mid \rho(x) \in K\} \\
&= X \backslash \{x \in X \mid \rho(x) \notin K\} \\
&= X \backslash \{x \in X \mid \rho(x) \in Y \backslash K\} \\
&= X \backslash (\rho^{-1}(Y \backslash K)) \\
&= (\rho^{-1}(K^c))^c
\end{aligned}$$

and since K is closed, K^c must be open, which means $\rho^{-1}(K^c)$ must be open (since ρ is continuous) which means $(\rho^{-1}(K^c))^c$ must be closed. So, $\rho^{-1}(K) = (\rho^{-1}(K^c))^c$ is closed and it is a subset of X which is a compact space. By Exercise L9-5, every closed subspace of a compact space is compact, so $\rho^{-1}(K)$ must be compact. Then, recalling that $U \subseteq Y$ was open, we have that $S(\rho^{-1}(K), Y)$ must be open by the definition of compact-open space on $\mathrm{Cts}(X, Y)$. Thus, $S(\rho^{-1}(K), U) \cap \operatorname{Im} \Psi$ is open in $\operatorname{Im} \Psi$, which shows that $\Psi(S(K, U))$ is open in $\operatorname{Im} \Psi$.

So, Ψ satisfies all three conditions of our lemma, hence it is a homeomorphism onto its image.

(ii)

Let X, Y be topological spaces with $X \neq \emptyset$, and let $\Psi\colon Y \to \text{Cts}(X,Y)$ be the map given by

$$\Psi(y) = c_y$$

where $c_y \in \text{Cts}(X,Y)$ is defined by $c_y(x) = y$ for all $x \in X$. We will show that Ψ is a homeomorphism onto its image by showing that Ψ satisfies the three conditions (continuous, injective, maps open sets in some sub-basis of Y to open sets of $\operatorname{Im}\Psi$) of the lemma we used for the previous two problem.

Ψ is continuous: The sets of the form $S(K,U)$ (where $K \subseteq X$ compact and $U \subseteq Y$ open) form a sub-basis for $\text{Cts}(X,Y)$, so to show Ψ is continuous it suffices to show that $\Psi^{-1}(S(K,U))$ is open in Y whenever $K \subseteq X$ is compact and $U \subseteq Y$ is open (Exercise L12-3). Indeed, let $K \subseteq X$ be compact and $U \subseteq Y$ be open. If $K = \emptyset$ then it is clear that $S(K,U) = \text{Cts}(X,Y)$ so that $\Psi^{-1}(S(K,U)) = \Psi^{-1}(\text{Cts}(X,Y)) = Y$, which is open. Otherwise if $K \neq \emptyset$ and we have

$$\begin{aligned} y \in \Psi^{-1}(S(K,U)) &\iff \Psi(y) \in S(K,U) \\ &\iff c_y \in S(K,U) \\ &\iff c_y(K) \subseteq U \\ &\iff y \in U \end{aligned}$$

so that $\Psi^{-1}(S(K,U)) = U$, and hence it is open. This proves that Ψ is continuous.

Ψ is injective: Suppose $y_1, y_2 \in Y$ such that $\Psi(y_1) = \Psi(y_2)$. Since $X \neq \emptyset$, pick and fix some $x_0 \in X$. Then

$$\Psi(y_1) = \Psi(y_2) \implies c_{y_1} = c_{y_2} \implies c_{y_1}(x_0) = c_{y_2}(x_0) \implies y_1 = y_2$$

and hence Ψ is injective.

Ψ maps open sets of Y to open sets of $\operatorname{Im}\Psi$: (Firstly, we'll just remark that Y is a sub-basis for itself, so if we prove that $\Psi(U)$ is open in $\operatorname{Im}\Psi$ for all open $U \subseteq Y$, the third dot point of our lemma will indeed be satisfied.) So now, let $U \subseteq Y$ be open, and fix some $x_0 \in X$ (which we can do since $X \neq \emptyset$), then we have

$$\Psi(U) = \{\Psi(y) \mid y \in U\} = \{c_y \mid y \in U\} = \operatorname{Im}\Psi \cap S(\{x_0\}, U).$$

Notice that $\{x_0\}$ is compact (one point space is compact), and we said earlier that U was open, hence $S(\{x_0\}, U)$ is open by the definition of compact-open space for $\text{Cts}(X,Y)$. Then, $\operatorname{Im}\Psi \cap S(\{x_0\}, U)$ is open in $\operatorname{Im}\Psi$ by the definition of subspace topology for $\operatorname{Im}\Psi \subseteq \text{Cts}(X,Y)$. Hence $\Psi(U)$ is open, as required.

Summary: We can now apply our lemma to conclude that Ψ is a homeomorphism onto its image. So we have proved that the map given in the question is continuous and a homeomorphism onto its image (even if X is not locally compact Hausdorff).