13 marks total
Notation and conventions

The set $\mathbb{N}$ is the set of positive integers $\{1, 2, 3, \ldots \}$.

The set $\mathbb{N}_0$ is the set of nonnegative integers $\{0, 1, 2, \ldots \}$.

The transpose of a matrix $X$ is denoted $X'$.
Question 1  

Lemma L2-2.  The pair \((S^1, d_a)\) is a metric space.  

Exercise L2-3.  Give a direct proof of Lemma L2-2 by dividing into cases as follows: Given \(x, y, z \in S^1\), set \(\theta := \Phi^{-1}(x), \theta' := \Phi^{-1}(y), \) and \(\theta'' := \Phi^{-1}(z)\).  Consider the following three statements:

\[
\begin{align*}
P_1 & : |\theta - \theta'| \leq \pi \\
P_2 & : |\theta' - \theta''| \leq \pi \\
P_3 & : |\theta - \theta''| \leq \pi
\end{align*}
\]

Each is either true or false for a particular triple \((x, y, z)\), and this means there are \(2^3 = 8\) cases (e.g. \(P_1, P_2\) true but \(P_3\) false).  Prove each case individually, and in this way prove the lemma.

Recall that \(d_a : S^1 \times S^1 \to \mathbb{R}_{\geq 0}\) is defined as

\[
d_a(\Phi(\omega), \Phi(\omega')) = \min\{|\omega - \omega'|, 2\pi - |\omega - \omega'|\} = \begin{cases} 
|\omega - \omega'|, & |\omega - \omega'| \leq \pi, \\
2\pi - |\omega - \omega'|, & |\omega - \omega'| > \pi.
\end{cases}
\]

for every \(\omega, \omega' \in [0, 2\pi)\).  We note that

(i) \(d_a\) is nonnegative: If \(\omega, \omega' \in [0, 2\pi)\) then \(0 \leq |\omega - \omega'| < 2\pi\) so that \(|\omega - \omega'|\) and \(2\pi - |\omega - \omega'|\) are both nonnegative;

(ii) \(d_a\) is symmetric: If \(\omega, \omega' \in [0, 2\pi)\) then \(|\omega - \omega'| = |\omega' - \omega|\);

(iii) \(d_a\) separates distinct elements: If \(\omega, \omega' \in [0, 2\pi)\) and \(d_a(\Phi(\omega), \Phi(\omega')) = 0\) then \(|\omega - \omega'| = 0\) (since \(2\pi - |\omega - \omega'| > 0\)) and \(\omega = \omega'\).  Conversely if \(\omega \in [0, 2\pi)\) then \(d_a(\Phi(\omega), \Phi(\omega)) = \min\{0, 2\pi\} = 0\).

To prove that \((S^1, d_a)\) is a metric space, it remains to establish that \(d_a\) satisfies the triangle equality: For our given \(x, y, z \in S^1\) we need to prove that

\[d_a(x, y) + d_a(y, z) \geq d_a(x, z)\]

We proceed by considering cases in the following order:

(a) \(P_1, P_2,\) and \(P_3\) are all false.

(b) \(P_1\) is true, while \(P_2\) and \(P_3\) are both false.

(c) \(P_2\) is true, while \(P_1\) and \(P_3\) are both false.

(d) \(P_1\) and \(P_2\) are true, while \(P_3\) is false.

(e) \(P_3\) is true, while \(P_1\) and \(P_2\) are both false.

(f) \(P_1\) and \(P_3\) are true, while \(P_2\) is false.

(g) \(P_2\) and \(P_3\) are true, while \(P_1\) is false.

(h) \(P_1, P_2,\) and \(P_3\) are all true.

For \(a, b \in \mathbb{R}\), we will use \(T(a, b)\) to denote the triangle inequality for \(|\cdot|\) in \(\mathbb{R}\), i.e. that \(|a| + |b| \geq |a + b|\).

1. \(\text{All false} \)
1(a) This case is not actually possible: Writing \([0, 2\pi]\) as the disjoint union \([0, \pi) \cup [\pi, 2\pi]\), the pigeonhole principle tells us that either \([0, \pi)\) or \([\pi, 2\pi]\) contains two of \(\theta, \theta', \text{ and } \theta''\). The difference of those two must therefore be strictly less than \(\pi\), so that \(P_1, P_2,\) and \(P_3\) cannot simultaneously all be false.

(b) \(P_1: P_2, P_3\) false

In this case we have
\[
d_a(x, y) + d_a(y, z) = |\theta - \theta'| + (2\pi - |\theta' - \theta''|),
\]
\[
d_a(x, z) = 2\pi - |\theta - \theta'|.
\]

Observe:
\[
d_a(x, y) + d_a(y, z) \geq d_a(x, z) \iff |\theta - \theta'| + (2\pi - |\theta' - \theta''|) \geq 2\pi - |\theta - \theta'|
\]
\[
\iff |\theta - \theta'| + |\theta - \theta''| \geq |\theta' - \theta''|
\]

The final inequality is precisely \(T(\theta' - \theta, \theta - \theta'')\), so we have shown that the triangle inequality holds for \(d_a\) when only \(P_1\) is true.

(c) \(P_2: P_1, P_3\) false

In this case we have
\[
d_a(x, y) + d_a(y, z) = (2\pi - |\theta - \theta'|) + |\theta' - \theta''|,
\]
\[
d_a(x, z) = 2\pi - |\theta - \theta''|.
\]

Observe:
\[
d_a(x, y) + d_a(y, z) \geq d_a(x, z) \iff (2\pi - |\theta - \theta'|) + |\theta' - \theta''| \geq 2\pi - |\theta - \theta'|\]
\[
\iff |\theta' - \theta''| + |\theta - \theta'| \geq |\theta' - \theta'|
\]

The final inequality is precisely \(T(\theta' - \theta'', \theta' - \theta)\), so we have shown that the triangle inequality holds for \(d_a\) when only \(P_2\) is true.

(d) \(P_1, P_2\) true; \(P_3\) false

In this case we have \(d_a(x, y) + d_a(y, z) = |\theta - \theta'| + |\theta' - \theta''|\). By \(T(\theta - \theta', \theta' - \theta'')\), this means that \(d_a(x, y) + d_a(y, z) \geq |\theta - \theta'|\). By definition, \(d_a(x, z) \leq |\theta - \theta''|\), so we have shown that the triangle inequality holds for \(d_a\) when \(P_1\) and \(P_2\) are true. (Note that we did not actually use the fact that \(P_3\) is false.)

(e) \(P_3\) true; \(P_1, P_2\) false

In this case we have
\[
d_a(x, y) + d_a(y, z) = (2\pi - |\theta - \theta'|) + (2\pi - |\theta' - \theta''|) = 4\pi - (|\theta - \theta'| + |\theta' - \theta''|),
\]
\[
d_a(x, z) = |\theta - \theta''|.
\]

Observe:
\[
d_a(x, y) + d_a(y, z) \geq d_a(x, z) \iff 4\pi - (|\theta - \theta'| + |\theta' - \theta''|) \geq |\theta - \theta''|
\]
\[
\iff |\theta - \theta'| + |\theta' - \theta''| + |\theta - \theta''| \leq 4\pi.
\]
Now, as a function of \((\theta, \theta', \theta'')\), the left-hand side of the last inequality is invariant under permutations, so without loss of generality we may assume that \(0 \leq \theta \leq \theta' \leq \theta'' < 2\pi\). The left-hand side then becomes

\[(\theta' - \theta) + (\theta'' - \theta') + (\theta'' - \theta) = 2(\theta'' - \theta) < 4\pi,\]

where the last inequality is because \(\theta'' - \theta < 2\pi\). Thus we have shown that the triangle inequality holds for \(d_a\) when only \(P_3\) is true.

\[(f) \quad P_1, P_3 \text{ true; } P_2 \text{ false}\]

In this case we have

\[d_a(x, y) + d_a(y, z) = |\theta - \theta'| + (2\pi - |\theta' - \theta''|),\]
\[d_a(x, z) = |\theta - \theta''|.

Observe:

\[d_a(x, y) + d_a(y, z) \geq d_a(x, z) \iff |\theta - \theta'| + (2\pi - |\theta' - \theta''|) \geq |\theta - \theta''| \]
\[\iff |\theta - \theta''| + |\theta' - \theta''| - |\theta - \theta'| \leq 2\pi \]
\[\iff |\theta - \theta''| + |\theta' - \theta''| - |\theta - \theta'| \leq 2|\theta - \theta''| \quad \text{since } |\theta - \theta''| \leq \pi \]
\[\iff |\theta' - \theta''| - |\theta - \theta'| \leq |\theta - \theta''| \]
\[\iff |\theta - \theta'| + |\theta - \theta''| \geq |\theta' - \theta''| \]
\[\iff |\theta' - \theta'| + |\theta - \theta''| \geq |\theta' - \theta''|.

The final inequality is precisely \(T(\theta' - \theta, \theta - \theta'')\), so we have shown that the triangle inequality holds for \(d_a\) when \(P_1\) and \(P_3\) are true but \(P_2\) is false.

\[(g) \quad P_2, P_3 \text{ true; } P_1 \text{ false}\]

In this case we have

\[d_a(x, y) + d_a(y, z) = (2\pi - |\theta - \theta'|) + |\theta' - \theta''|,\]
\[d_a(x, z) = |\theta - \theta''|.

Observe:

\[d_a(x, y) + d_a(y, z) \geq d_a(x, z) \iff (2\pi - |\theta - \theta'|) + |\theta' - \theta''| \geq |\theta - \theta''| \]
\[\iff |\theta - \theta''| + |\theta' - \theta''| - |\theta - \theta'| \leq 2\pi \]
\[\iff |\theta - \theta''| + |\theta' - \theta''| - |\theta - \theta'| \leq 2|\theta - \theta''| \quad \text{since } |\theta - \theta''| \leq \pi \]
\[\iff |\theta - \theta'| - |\theta' - \theta''| \leq |\theta - \theta''| \]
\[\iff |\theta - \theta'| - |\theta' - \theta''| \leq |\theta - \theta''| \]
\[\iff |\theta - \theta'| + |\theta' - \theta''| \geq |\theta - \theta'| \]
\[\iff |\theta - \theta'| + |\theta' - \theta''| \geq |\theta - \theta'|.

The final inequality is precisely \(T(\theta - \theta'', \theta' - \theta)\), so we have shown that the triangle inequality holds for \(d_a\) when \(P_2\) and \(P_3\) are true but \(P_1\) is false.

\[(h) \quad \text{All true}\]

Please refer to the case where only \(P_1\) and \(P_2\) are true.
Question 2  \( (3 \text{ marks}) \)

Exercise L3-3.  Prove that any element of \(\text{Isom}(S^1, d_a)\) of the form

\[
F = g_1g_2\ldots g_r, \quad r \geq 0,
\]

where each \(g_i\) is either \(R_\theta\) for some \(\theta \in \mathbb{R}\) or \(T\), may be proven equal to \(R_\psi T^n\) for some \(\psi \in [0, 2\pi)\) and \(n \in \{0, 1\}\), using relations (R1), (R2), and (R3).

Recall that the relations were (R1) \(R_\theta R_\phi = R_{\theta + \phi}\), (R2) \(R_\theta T = TR_{-\theta}\), and (R3) \(T^2 = \text{id}\).

We proceed by induction. For each \(m \in \mathbb{N}_0\), let \(P(m)\) be the following proposition: For every \(m\)-tuple \((g_1, g_2, \ldots, g_m)\) where each \(g_i\) is either \(R_\theta\) or \(T\), there exist \(\psi \in [0, 2\pi)\) and \(n \in \{0, 1\}\) such that \(g_1g_2\ldots g_m = R_\psi T^n\).

Note that \(P(0)\) is simply the proposition that \(\text{id} \in \text{Isom}(S^1, d_a)\) may be written as \(R_\psi T^n\) for some \(\psi \in [0, 2\pi)\) and \(n \in \{0, 1\}\). Taking \(\psi := 0\) and \(n := 0\), we see that \(P(0)\) holds.

Suppose \(k \in \mathbb{N}_0\) is such that \(P(k)\) holds. We will show that \(P(k + 1)\) holds. For a given \((g_1, g_2, \ldots, g_{k+1})\), we wish to show that there exist \(\psi \in [0, 2\pi)\) and \(n \in \{0, 1\}\) such that \(g_1g_2\ldots g_{k+1} = R_\psi T^n\). Since \(P(k)\) holds, there exist \(\omega \in [0, 2\pi)\) and \(p \in \{0, 1\}\) such that \(g_1g_2\ldots g_k = R_\omega T^p\). Hence it is sufficient to show that there exist \(\psi \in [0, 2\pi)\) and \(n \in \{0, 1\}\) such that \(R_\omega T^p g_{k+1} = R_\psi T^n\).

If \(g_{k+1}\) is \(R_\theta\) for some \(\theta \in \mathbb{R}\) then, using (R2) \(p\) times, we have

\[
T^p g_{k+1} = T^p R_\theta = \begin{cases} R_\theta, & p = 0, \\
R_{-\theta}T, & p = 1, \end{cases}
\]

which we may write as \(R_{(-1)^p \theta} T^p\). Hence \(R_\omega T^p g_{k+1} = R_\omega R_{(-1)^p \theta} T^p\). Using (R1), we have \(R_\omega T^p g_{k+1} = R_{\omega + (-1)^p \theta} T^p\). Finally, taking \(\psi \) to be \((\omega + (-1)^p \theta) \mod 2\pi\) and \(n := p\), we have

\[
g_1g_2\ldots g_k g_{k+1} = R_{\omega + (-1)^p \theta} T^p = R_\psi T^n.
\]

If \(g_{k+1}\) is \(T\) then using (R3)

\[
T^p g_{k+1} = T^p T = \begin{cases} T, & p = 0, \\
\text{id}, & p = 1, \end{cases}
\]

which we may write as \(T^{1-p}\). Hence \(R_\omega T^p g_{k+1} = R_\omega T^{1-p}\). Taking \(\psi := \omega\) and \(n := 1 - p\), we have

\[
g_1g_2\ldots g_k g_{k+1} = R_\omega T^{1-p} = R_\psi T^n.
\]

In both cases we have produced \(\psi \in [0, 2\pi)\) and \(n \in \{0, 1\}\) such that \(R_\omega T^p g_{k+1} = R_\psi T^n\). Thus, we have shown that \(P(k + 1)\) holds if \(P(k)\) holds. By the principle of mathematical induction, we therefore have that \(P(m)\) holds for every \(m \in \mathbb{N}_0\).
Let \( \ell \) denote the line passing through the origin and \((\cos(\theta/2), \sin(\theta/2))\).

Since \( R_\theta \), \( T \), and reflection in \( \ell \) are all linear transformations, it suffices to show that the image of \((1,0)\) and \((0,1)\) under \( R_\theta T \) are their respective images under reflection in \( \ell \).

Let us first determine these images: The (directed and counterclockwise) angles subtended at the origin are

- from \((1,0)\) to \((\cos(\theta/2), \sin(\theta/2))\): \(\theta/2\), and
- from \((0,1)\) to \((\cos(\theta/2), \sin(\theta/2))\): \(-\pi - \theta)/2\).

Thus, the image of \((1,0)\) should be a counterclockwise rotation of \((\cos(\theta/2), \sin(\theta/2))\) around the origin by \(\theta/2\), while the image of \((1,0)\) should be a counterclockwise rotation of \((\cos(\theta/2), \sin(\theta/2))\) around the origin by \(-\pi - \theta)/2\). That is, \((1,0)\) is sent to \((\cos \theta, \sin \theta)\), and \((0,1)\) is sent to

\[
(\cos(\theta/2 - \pi - \theta/2), \sin(\theta/2 - \pi - \theta/2)) = (\cos(\theta - \pi), \sin(\theta - \pi/2)) = (\sin \theta, -\cos \theta).
\]

The images of \((1,0)\) and \((0,1)\) under \( R_\theta T \) are given by

\[
\begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
0 & -1
\end{bmatrix}
\begin{bmatrix}
1 \\
0
\end{bmatrix}
\text{ and }
\begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
0 & -1
\end{bmatrix}
\begin{bmatrix}
0 \\
1
\end{bmatrix}
\begin{bmatrix}
\sin \theta \\
-\cos \theta
\end{bmatrix}
\]

respectively. These agree with their images under reflection in \( \ell \). Hence the linear transformations \( R_\theta T \) and reflection through the line passing through the origin and \((\cos(\theta/2), \sin(\theta/2))\) are the same transformations.
Question 4  \( (5 \text{ marks}) \)

Exercise L3-5.  Let \( F : V \to V \) be an invertible linear operator on a finite-dimensional vector space.

(a) Prove that precisely one of the following two possibilities is realised:

(I) \( \forall B \ (F(B) \sim B) \)

(II) \( \forall B \ (F(B) \not\sim B) \)

where \( B \) ranges over all ordered bases and \( F(B) \) denotes \( (F(b_1), \ldots, F(b_n)) \) if \( B = (b_1, \ldots, b_n) \). In the first case we say \( F \) is orientation preserving, and in the latter case we say \( F \) is orientation reversing.

(b) Prove that \( F \) is orientation preserving iff \( \det(F) > 0 \) and orientation reversing iff \( \det(F) < 0 \).

(c) Define

\[
O(n) := \{ X \in M_n(\mathbb{R}) \mid X \text{ is orthogonal, i.e. } X'X = I_n \} \\
SO(n) := \{ X \in O(n) \mid \det(X) = 1 \}.
\]

Prove that \( X \in O(n) \) if and only if for all \( v, w \in \mathbb{R}^n \)

\[
(Xv) \cdot (Xw) = v \cdot w.
\]

By part (b), \( SO(n) \) are precisely the matrices in \( O(n) \) that give rise to orientation preserving linear transformations \( \mathbb{R}^n \to \mathbb{R}^n \).

(d) Prove that \( O(n) \) is a group under multiplication, and \( SO(n) \) is a subgroup. Produce an element \( T \in O(n) \) such that \( T^2 = \text{id} \) and every element of \( O(n) \) not in \( SO(n) \) may be written as \( XT \) for some \( X \in SO(n) \). Thus prove \( SO(n) \subseteq O(n) \) is a normal subgroup and that there is a group isomorphism

\[
O(n)/SO(n) \cong \mathbb{Z}/2\mathbb{Z}.
\]

(a) Fix an ordered basis \( C \) of \( V \). Since \( F \) is a linear operator on a finite-dimensional vector space, for some matrix \( A \in M_n(\mathbb{R}) \) (where \( n = \dim V \)) we have \( F(C) = AC \). Note that because \( F \) is invertible, \( A \) must be an invertible matrix. That is, \( \det A \neq 0 \).

Now, our definition of \( A \) above means that \( A = [\text{id}]^C_{F(C)} \), where \( [\text{id}]^C_{F(C)} \in M_n(\mathbb{R}) \) is the matrix which changes \( F(C) \)-coordinates to \( C \)-coordinates. For our basis \( C \) we have by definition (from Tutorial 1)

\[
F(C) \sim C \quad \iff \quad \det A > 0.
\]

By negating (and noting that \( \det A \neq 0 \) as mentioned above) we also have for our basis \( C \) that

\[
F(C) \not\sim C \quad \iff \quad \det A < 0.
\]

Next, consider another arbitrary ordered basis \( B \) of \( V \). Then \( [\text{id}]^B_{F(B)} \) is the matrix which changes \( F(B) \)-coordinates to \( B \)-coordinates. We can see that this matrix is the same as \( [F]_B^B \), the matrix corresponding to applying \( F \) under \( B \)-coordinates. We now carry out a change of basis to relate \( [F]_B^B \) and \( [F]_C^C = [\text{id}]_C^{F(C)}A[\text{id}]_B^{F(B)} \):

We write

\[
[F]_B^B = [\text{id}]_B^C [F]_B^C [\text{id}]_B^B = [\text{id}]_C^C A [\text{id}]_B^B.
\]

From this we can see that

\[
F(B) \sim B \quad \iff \quad \det([\text{id}]_B^B) > 0 \quad \iff \quad \det([\text{id}]_C^C A [\text{id}]_B^B) > 0.
\]
We simplify the last condition by writing
\[ \det([id]_B^B A [id]_B^B) = \det([id]_B^B)(\det A) \det([id]_B^C), \]
\[ = (\det A) \det([id]_B^B) \det([id]_B^C), \]
\[ = \det A \text{ since } [id]_C^B [id]_B^C = [id]_B^B = I_n. \]

This allows us to conclude that
\[ F(B) \sim B \iff \det(A) > 0. \]

Since \( B \) was an arbitrary ordered basis of \( V \), we have in fact established that
\[ F(B) \sim B \quad \forall \text{ bases } B \text{ of } V \iff \det A > 0, \]
as well as
\[ F(B) \not\sim B \quad \forall \text{ bases } B \text{ of } V \iff \det A < 0. \]

by negating. Precisely one of \( \det A > 0 \) and \( \det A < 0 \) is true, so we may conclude that either (I) \( F(B) \sim B \)
for every basis \( B \) of \( V \), or (II) \( F(B) \not\sim B \) for every basis \( B \) of \( V \).

(b) In the previous part we proved that it makes sense to classify the invertible linear operator \( F \) itself as being orientation-preserving or orientation-reversing, since applying \( F \) either preserves the orientation of all bases or preserves the orientation of no bases at all. As demonstrated above, since \( \det F \neq 0 \), \( F \) is orientation-preserving if and only if \( \det F > 0 \), while \( F \) is orientation-reversing if and only if \( \det F < 0 \).

(c) First observe that for every \( X \in M_n(\mathbb{R}) \) and for every \( v, w \in \mathbb{R}^n \) we have
\[ (Xv) \cdot (Xw) = (Xv)'(Xw) = v'X'Xw. \]

(\( \Rightarrow \)) Suppose \( X \in O(n) \). Then \( X'X = I_n \), so for every \( v, w \in \mathbb{R}^n \) we have
\[ (Xv) \cdot (Xw) = v'I_n w = v \cdot w. \]

(\( \Leftarrow \)) Suppose that \( (Xv) \cdot (Xw) = v \cdot w \) for every \( v, w \in \mathbb{R}^n \). For every \( i, j \in \{1, 2, \ldots, n\} \), let \( e_i \in \mathbb{R}^n \) denote the column vector consisting of 1 as the \( i \)th coordinate and 0 in every other coordinate. Then for every \( i, j \in \{1, 2, \ldots, n\} \) we have
\[ (Xe_i) \cdot (Xe_j) = e_i'X'Xe_j = (X'X)_{i,j}, \]
where \( (X'X)_{i,j} \) is the \( (i, j) \)th entry of \( X'X \). By hypothesis, this must be equal to \( e_i \cdot e_j = 1(i = j) \). That is, the \( (i, j) \)th entry of \( X'X \) must be 1 if \( i = j \) and 0 if \( i \neq j \). Thus \( X'X = I_n \) and \( X \in O(n) \).

(d) We begin by showing that \( O(n) \) is a group under multiplication. The identity element is \( I_n \) (which is orthogonal).

We claim that the inverse of \( X \in O(n) \) is \( X' \in O(n) \). Recall that \( X \) and \( X' \) are inverses in the group \( GL(n) \) if \( X \) is orthogonal (i.e. \( X'X = XX' = I_n \)), so it remains to show that \( X' \in O(n) \). Observe that \( X''X' = (X'X)' = (XX^{-1})' = I_n \), so indeed \( X' \in O(n) \).

Next, if \( X, Y \in O(n) \) then
\[ (XY)'(XY) = Y'(X'X)Y = Y'Y = I_n, \]
so that \( XY \in O(n) \). This shows that \( O(n) \) is a group under matrix multiplication.

To show that \( SO(n) \) is a subgroup of \( O(n) \), we need to show that (i) \( SO(n) \) contains the identity \( I_n \); (ii) \( SO(n) \) is closed under inversion; and (iii) \( SO(n) \) is closed under matrix multiplication.
Since \( \det I_n = 1 \), we know that \( I_n \in SO(n) \), so \( SO(n) \) contains the identity. Next, if \( X \in O(n) \) and \( \det X = 1 \) then \( X' \in O(n) \) and \( \det(X') = \det(X^{-1}) = (\det X)^{-1} = 1 \), so \( X' \in SO(n) \). Hence \( SO(n) \) is closed under inversion. Finally, if \( X, Y \in O(n) \) and \( \det X = \det Y = 1 \), then \( XY \in O(n) \) and \( \det(XY) = (\det X)(\det Y) = 1 \), so \( XY \in SO(n) \). Thus, \( SO(n) \) is a subgroup of \( O(n) \).

We claim that a suitable choice of \( T \in O(n) \) is

\[
\begin{pmatrix}
-1 & 0 & \ldots & 0 \\
0 & 1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 1
\end{pmatrix}
\]

the \( n \times n \) diagonal matrix with \(-1\) in the upper left entry and \(1\) everywhere else along the diagonal. We verify that \( T \) is orthogonal: Note that \( T' = T \) and \( T^2 = I_n \), so that \( T'T = T^2 = I_n \) and \( T \in O(n) \). In our discussion below we will use the fact that \( \det T = -1 \).

Next, consider \( Y \in O(n) \). Both \( Y \) and \( YT \) are elements in \( O(n) \). Since \( Y'Y = I_n \), we know that \((\det Y)^2 = 1\), so that \( \det Y = 1 \) or \( \det Y = -1 \). If \( Y \notin SO(n) \), we necessarily have that \( \det Y = -1 \), so that \( \det(YT) = (\det Y)(\det T) = 1 \) and \( YT \in SO(n) \). With the choice of \( X := YT \), we have \( X \in SO(n) \) and \( XT = YT^2 = Y \). Thus our \( T \) satisfies the required criteria.

Before showing that \( SO(n) \) is a normal subgroup of \( O(n) \), we will first show that \( T \) in fact satisfies a further similar property. In particular, we will show that if \( Y \in O(n) \setminus SO(n) \), then \( Y = TZ \) for some \( Z \in SO(n) \): We simply take \( Z := TY \in O(n) \) and note that \( \det Y = -1 \) since \( Y \in O(n) \setminus SO(n) \), so that \( \det Z = (\det T)(\det Y) = 1 \) and \( Z \in SO(n) \). Then we verify that \( TZ = T^2Y = Y \).

The above results show that the right cosets of \( O(n) \) with respect to \( SO(n) \) is the set \( \{ SO(n), SO(n)T \} \), while the left cosets of \( O(n) \) with respect to \( SO(n) \) is the set \( \{ SO(n), T SO(n) \} \). Since each is a bipartition of \( SO(n) \), we have that \( SO(n)T = T SO(n) \).

We are now ready to show normality of \( SO(n) \) as a subgroup of \( O(n) \): If \( Y \in SO(n) \) then \( Y SO(n) = SO(n) = SO(n)Y \). If \( Y \in O(n) \setminus SO(n) \) then for some \( X \in SO(n) \) and \( Z \in SO(n) \) we have \( Y = XT = TZ \) and

\[
Y SO(n) = TZ SO(n) = T SO(n) = SO(n)T = SO(n)XT = SO(n)Y.
\]

This shows that \( SO(n) \) is a normal subgroup of \( O(n) \). Since there are only two cosets, the quotient group \( O(n)/SO(n) \) is a group of order 2, so \( O(n)/SO(n) \cong \mathbb{Z}/2\mathbb{Z} \).
**Question 5**

**Exercise L4-0.** Prove that

\[
d_1(x, y) = \sum_{i=1}^{n} |x_i - y_i| \\
d_\infty(x, y) = \max\{|x_i - y_i|\}_{i=1}^{n}
\]

define metrics on \(\mathbb{R}^n\).

Since \(|\cdot|\) is nonnegative, and the sum and the maximum of nonnegative numbers are nonnegative, both \(d_1\) and \(d_\infty\) are nonnegative. Furthermore, since \(|a - b| = |b - a|\) for all \(a, b\) we have \(d_1(x, y) = d_1(y, x)\) and \(d_\infty(x, y) = d_\infty(y, x)\) for all \(x, y \in \mathbb{R}^n\). That is, both \(d_1\) and \(d_\infty\) are symmetric.

It remains to show that \(d_1\) and \(d_\infty\) both separate distinct points and both satisfy the triangle inequality.

Let us proceed first for \(d_1\). Suppose \(d_1(x, y) = 0\). Since each \(|x_i - y_i|\) is nonnegative, the only way for the sum \(\sum_{i=1}^{n} |x_i - y_i|\) to be 0 is to have \(|x_i - y_i|\) be exactly 0 for every \(i\). That is, we must have \(x_i = y_i\) for every \(i\), so that \(x = y\). Hence \(d_1\) separates distinct points. For the triangle inequality, fix \(x, y, z \in \mathbb{R}^n\). Then

\[
d_1(x, y) + d_1(y, z) = \sum_{i=1}^{n} |x_i - y_i| + \sum_{i=1}^{n} |y_i - z_i| = \sum_{i=1}^{n} (|x_i - y_i| + |y_i - z_i|) \geq \sum_{i=1}^{n} |x_i - z_i|,
\]

where the inequality comes from the triangle inequality for \(|\cdot|\) on \(\mathbb{R}\). The last term is precisely \(d_1(x, z)\), so we have established that \(d_1(x, y) + d_1(y, z) \geq d_1(x, z)\). Therefore \(d_1\) satisfies the triangle inequality, meaning that \(d_1\) meets all the conditions of being a metric on \(\mathbb{R}^n\).

We now turn to \(d_\infty\). Suppose \(d_\infty(x, y) = 0\). Then for every \(i\) we must have \(|x_i - y_i| \leq 0\), so that \(x_i = y_i\). Hence \(x = y\) if \(d_\infty(x, y) = 0\), and \(d_\infty\) separates distinct points. For the triangle inequality, fix \(x, y, z \in \mathbb{R}^n\). Observe that for every \(i\) we necessarily have

\[
d_\infty(x, y) + d_\infty(y, z) \geq |x_i - y_i| + |y_i - z_i| \geq |x_i - z_i|,
\]

where the first inequality comes from the definition of \(d_\infty\) and the second inequality comes from the triangle inequality for \(|\cdot|\) on \(\mathbb{R}\). As this holds for every \(i\), we may conclude that

\[
d_\infty(x, y) + d_\infty(y, z) \geq \max\{|x_i - z_i|\}_{i=1}^{n} = d_\infty(x, z).
\]

Hence \(d_\infty\) satisfies the triangle inequality, and \(d_\infty\) too meets all the conditions of being a metric on \(\mathbb{R}^n\).
Question 6

Exercise L4-4. Prove that if $P_1 = Q^{-1}P_2Q$ for some orthogonal matrix $Q$ then multiplication by $Q$ gives an isometry (assume $P_1, P_2$ positive-definite)

$$(\mathbb{R}^n, d_{P_1}) \longrightarrow (\mathbb{R}^n, d_{P_2})$$

That is, the metric we get on $\mathbb{R}^n$ from $P_1$ is essentially the same as the one we get from $P_2$.

Since $Q$ is orthogonal, $Q$ is invertible, and multiplication by $Q$ is a bijection from $\mathbb{R}^n$ to $\mathbb{R}^n$ (in particular, it is surjective). It remains to show that multiplication preserves distance. Fix $x, y \in \mathbb{R}^n$ (treated as column vectors). We wish to show that $d_{P_2}(Qx, Qy) = d_{P_1}(x, y)$. We write

$$d_{P_2}(Qx, Qy) = (Qx)'P_2(Qy) = x'(Q'P_2Q)y.$$ 

Since $Q$ is orthogonal, we have $Q' = Q^{-1}$, so that $Q'P_2Q = Q^{-1}P_2Q = P_1$, and

$$d_{P_2}(Qx, Qy) = x'P_1y = d_{P_1}(x, y).$$

This shows that multiplication by $Q$ is distance-preserving. As mentioned, multiplication by $Q$ is also surjective, so altogether multiplication by $Q$ is an isometry from $(\mathbb{R}^n, d_{P_1})$ to $(\mathbb{R}^n, d_{P_2})$. 