MAST30026Assignment1

Yanchao Yang
Tutor: Daniel Murfet
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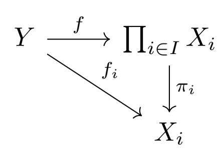
1 L7-5.5 (box product)

Given topological spaces $\{X_i\}_{i\in I}$, let $\prod_{i\in I}^{alt} X_i$ denote the set $\prod_{i\in I} X_i$ with the alternate topology which has a basis the sets $\prod_{i\in I} U_i$ where $U_i \subseteq X_i$ is open for all $i \in I$. (i.e. we do not impose the condition that $\{i \in I | U_i \neq X_i\}$ is finite). Prove that this is a valid basis, but give a counterexample to show $\prod_{i\in I}^{alt} X_i$ does not have the universal property of Lemma L7-2.

Background story: The basis of a product space is,

$$\beta = \left\{\prod_{i\in I} U_i | \forall i \in I, U_i \subseteq X_i \text{ is open and } Q \text{ is finite}\right\}$$

where $Q = \{i \in I | U_i \neq X_i\}$ and $I = Q \cup (I \setminus Q)$. And this basis helps us to establish the so-called Universal Property of the Product.



The three actors in this Universal Property are three functions $f \in \text{Cts}\left(Y, \prod_{i\in I} X_i\right)$, $f_i \in \text{Cts}\left(Y, X_i\right)$ and $\pi_j : \prod_{i\in I} X_i \to X_j$ which is called j-projection and defined as $\pi_j\left((x_i)_{i\in I}\right) = x_j$ for some $(x_i)_{i\in I} \in \prod_{i\in I} X_i$. In fact, from the Exercise L7-5 that the j-projection is continuous. The full statement is: given f_i that is continuous, there exists unique f which is defined as $\forall i \in I, \pi_i \circ f = f_i$ which is continuous.

Sketch of proof: Define the basis for the alternative topology to be $\prod_{i\in I}^{alt} X_i$ is β^{alt} which is defined by

$$\beta^{alt} = \left\{\prod_{i\in I} U_i | \forall i \in I, U_i \subseteq X_i \text{ is open}\right\}$$

The proof has two parts. First part is showing that β^{alt} is indeed a basis. This is done by checking the condition

$$\forall U \in \prod_{i\in I} X_i, \forall x \in U, \exists B \in \beta^{alt}, (x \in B \land B \subseteq U)$$

We will check an equivalent condition that every U open in the product topology can be written as the union set of a subset $\mathcal{C}^{alt} \subseteq \beta^{alt}$.

The second part is to show it fails Universal Property of the Product. The idea is to construct a counter example. Assume the topological space $X = \prod_{i\in I} X_i$ is also a metric space (X, d) with an arbitrary metric d. In fact, let's assume $d = d_2$. Further assume Y and all X_i's are equal to the real line $\mathbb{R}$, the unknown indexed family I equals to the natural number $\mathbb{N}$. By this assumption, we can see the basis will be a set of open balls $\{B_\epsilon(x) \,|x \in X, \epsilon > 0\} =: \beta^{alt}$. The setup is now complete. The strategy of the proof is by using proof by contradiction. Given f_i be the identity function id_i for all $i \in \mathbb{N}$, and assume such continuous function f exists. Pick an arbitrary open set U from $\prod_{i\in\mathbb{N}} \mathbb{R} = \mathbb{R}^{\mathbb{N}}$, consider the preimage of the intersection of these U's under f. Since U's is open, it can be represented as a union of open balls. And at some stage, we can see the preimage of this arbitrary open set U can be represented by an infinite intersection of open balls. We know this intersection of open balls will shrink to a point, which is NOT open in the metric setting. Hence by contradiction, f cannot be continuous, and the Universal Property of the Product failed.

1.1 Prove β^{alt} is a basis

Proof. <u>WTS:</u> $\forall U \subseteq \prod_{i\in I} X_i$ open $, \exists \mathcal{C}^{alt} \subseteq \beta^{alt}, U = \bigcup\limits_{B\in\mathcal{C}^{alt}} B$.
Pick an arbitrary $U \in \prod_{i\in I} X_i$ open in the product topology.
i.e. $U = \prod_{i\in I} U_I$ for some $U_i \in \mathcal{T}_{X_i}$.
$\implies U \in \beta^{alt} = \left\{\prod_{i\in I} U_i | \forall i \in I, U_i \subseteq X_i \text{ is open}\right\}$
We can use such $\mathcal{C}^{alt}$ to be a set just contain U, $\mathcal{C}^{alt} = \{U\}$. Of course $U = \bigcup_{B\in\mathcal{C}^{alt}} B$. Hence such $\mathcal{C}^{alt} \subseteq \beta^{alt}$ exists. Therefore, β^{alt} is indeed a basis for the product topology. $\square$

1.2 Prove the Universal Property of the Product failed

Proof. Make the following assumptions:

1. Assume $X_i = \mathbb{R}$ for all $i \in I$ and $Y = \mathbb{R}$ also be a metric space with metric d. Hence, each $X_i = \mathbb{R}$ has the topology $\mathcal{T}_d$.
2. Assume $f_i = id_i$ for all $i \in I$ to be the identity function. Identity function $id_i : \mathbb{R} \to \mathbb{R}$ is course continuous, so $id_i \in \text{Cts}(\mathbb{R}, \mathbb{R})$.
3. Let the indexed set I be the natural numbers $\mathbb{N}$

Assume by contradiction that the Universal Property of the Product hold under the basis β^{alt}. Therefore, if we have a function $f : \mathbb{R} \to \prod_{n\in\mathbb{N}} \mathbb{R}_n$,

then by Universal Property of the Product, such bijection of j-projection $\pi_j : \prod_{n\in\mathbb{N}} \mathbb{R}_n \to \mathbb{R}$ which make $\pi_j \circ f = id_j$ exist for all $j \in \mathbb{N}$. And f should also be continuous. We can define such inverse of π_j to have $f(y) = (id_n)_{n\in\mathbb{N}}$.
Pick an arbitrary $U \subseteq \mathbb{R}^{\mathbb{N}}$ be open in product topology generated by the basis β^{alt}.
i.e. $U = \prod_{n\in\mathbb{N}} U_n$ for some $U_n \in \mathcal{T}_d$.
Because $U_n \in \mathcal{T}_d$, and the basis of the topology $\mathcal{T}_d$ is a set of open balls $B_\epsilon(x) = \{y \in \mathbb{R} | d(x,y) < \epsilon\}$ for some $x \in U_n$ and $\epsilon > 0$.
i.e. $U_n = \bigcup_{x\in U_n} B_\epsilon(x)$
In fact, we can choose U have some $U_n = B_\epsilon(x)$ and we just focus on a single point $x = 0$ and consider the product of open balls around it.
Choose $\epsilon = \frac{1}{n}$, so that our open ball $B_{\frac{1}{n}}(0) = \left(-\frac{1}{n}, \frac{1}{n}\right)$ will shrink as $n \in \mathbb{N}$ grows.
Consider $f^{-1}U$, which is open in $\mathbb{R}$ by our hypothesis:

$$\begin{aligned}
f^{-1}U &= f^{-1}\left(\prod_{n\in\mathbb{N}} U_n\right) \\
&= \left\{y \in \mathbb{R} | f(y) \in \prod_{n\in\mathbb{N}} U_n\right\} \\
&= \left\{y \in \mathbb{R} | (id_n(y))_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} U_n\right\} \\
&= \{y \in \mathbb{R} | \forall n \in \mathbb{N}, y \in U_n\} \\
&= \bigcap_{n\in\mathbb{N}} U_n \\
&= \bigcap_{n\in\mathbb{N}} \bigcup_{x\in U_n} B_{\frac{1}{n}}(x) \\
&= \bigcap_{n\in\mathbb{N}} B_{\frac{1}{n}}(0) \\
&= \bigcap_{n\in\mathbb{N}} \left(-\frac{1}{n}, \frac{1}{n}\right) \\
&= \{*\}, \text{ where } \{0\} \text{ indicates a point set}
\end{aligned}$$

The last line is because as $n \to \infty$, no matter which point $x \in U_n$ we are considering, the distance between x and points around it will shrink to 0. i.e. $d(x,y) < \frac{1}{n} \to 0$ for all $y \in \mathbb{R}$ as $n \to \infty$.

But a point set $\{0\}$ is NOT open in $\mathbb{R}$ Because it contains no open balls around it. Hence we have a contradiction. By proof of contraction, f is not continuous and hence the Universal Property of the Product has failed for the basis β^{alt}.

$\square$

2 L7-17 (continuous components)

Given spaces $\{X_i\}_{i\in I}$ and Y, prove a function $Y \xrightarrow{f} \prod_{i\in I} X_i$ with components $f_i : Y \to X_i$ is continuous iff f_i is continuous for all $i \in I$

Background story: We have a function $f : Y \to \prod_{i\in I} X_i$ with its components, say $f_i : Y \to X_i$. So that we can write f as a tuple of its components $f = (f_i)_{i\in I}$. For example, if we use f to map a $y \in Y$ into $\prod_{i\in I} X_i$, we can write

$$f(y) = (f_i(y))_{i\in I}$$

Note that there is no implies order of this tuple because the indexed set I is arbitrary.

We also make use of the product topology, which a topology generated by the basis

$$\beta = \left\{\prod_{i\in I} U_i | \forall i \in I, U_i \subseteq X_i \text{ is open and } Q \text{ is finite}\right\}$$

where $Q = \{i \in I | U_i \neq X_i\}$ and $I = Q \cup (I \setminus Q)$.

Sketch of proof: We will prove in the first part of the forward direction

$$f \in \text{Cts}\left(Y, \prod_{i\in I} X_i\right) \implies f_i \in \text{Cts}(Y, X_i)$$

by using fact that the preimage of all open sets in $\prod_{i\in I} X_i$ are open in Y. In execution, we will choose an arbitrary open set $V \in \prod_{i\in I} X_i$ [1], and we will focus on a particular $j \in I$. Without loss of generality, we can choose the open set of interest V to be an arbitrary set from the basis β. We write such open as $V = U$ to identify it is from the basis. In addition, because we are only interested in that particular $j \in I$, we will choose the set Q to be $Q = \{j | j \in I \text{ for } U_j = X_j\}$. This U_j will be the arbitrary choice from $\mathcal{T}_{X_j}$. The consequence is you can imagine that if we pick an arbitrary $U \in \beta$, it will has the form

$$U = \left(\prod_{i\in I\setminus\{j\}} X_j\right) \times U_j = \ldots X_{j-1} \times U_j \times X_{j+1} \times \ldots$$

Then at some point, you can find the set $f^{-1}(U)$ will become $f_i^{-1}(U)$ which is the set of interest.

And prove the backward direction in the second part

$$f_i \in \text{Cts}(Y, X_i) \implies f \in \text{Cts}\left(Y, \prod_{i\in I} X_i\right)$$

Just pick an arbitrary U from the basis of product topology. Proceed will get the result.

[1] Note here, $V = \prod_{i\in I} V_i$ for some $V_i \subseteq X_i$ open in X_i for all $i \in I$. Because all elements in the product topology are generated by the basis β, and all elements $U = \prod_{i\in I} U_i \in \beta$ has its components U_i open in X_i. So if we pick an element, say $V = \prod_{i\in I} V_i$ from the product topology, each V will be an arbitrary union or finite intersection of U's, hence each V_i will also be some arbitrary union or finite intersection of U_i's. We know topology is closed under finite intersection and arbitrary unions. Therefore, we know each V_i will also be open in each X_i for all $i \in I$.

2.1 Forward direction

Proof. Assume f is continuous.
i.e. $\forall U \subseteq \prod_{i\in I} X_i U$ **open in** $\prod_{i\in I} X_i \implies f^{-1}U \in \mathcal{Y}$.
Pick one aribrary $U_j \in \mathcal{T}_{X_j}$. Choose a $U \subseteq \prod_{i\in I} X_i$ that has its j^{th} component equal to U_j and other component equal to X_k forall $k \in I \setminus \{j\}$. By definition of product topology, such U is open in product topology, more precisely, the product topology with the basis has $Q = \{j | j \in I \text{ for } U_j = X_j\}$ and the indexed family $I = Q \cup I \setminus Q$.
Consider $f^{-1}U \in \mathcal{T}_Y$:

$$\begin{aligned}
f^{-1}U &= \{y \in Y | fy \in U\} \\
&= \left\{ y \in Y | fy \in \prod_{i\in I\setminus\{j\}} X_i \cup U_j \right\} \\
&= \left\{ y \in Y | \left(f_i y\right)_{i\in I} \in \prod_{i\in I\setminus\{j\}} X_i \cup U_j \right\} \\
&= \{y \in Y | \forall i \in I \setminus \{j\}\, f_i y \in X_i \wedge f_j y \in U_j\} \\
&= \left\{ y \in Y | \forall i \in I \setminus \{j\}\, y \in \underbrace{f_i^{-1} X_i}_{=Y} \wedge y \in f_i^{-1} U_j \right\} \\
&= \left\{ y \in Y | y \in f_i^{-1} U_j \right\} \\
&= f_i^{-1} U_j \\
&\in \mathcal{T}_Y
\end{aligned}$$

Hence, we proved U_j open in X_i implies $f_i^{-1}U_j$. Since the choice of j and U_j are all arbitrary, so we can infer f_i are continuous $\forall i \in I$. □

2.2 Backward direction

Proof. Assume $\forall i \in I, f_i \in \text{Cts}(Y, X_i)$.
i.e. $\forall U_i \subseteq, (U_i \in \mathcal{T}) X_i \implies f_i^{-1} U_i \in \mathcal{T}_Y$.
Pick an $U \subseteq \prod_{i\in I} X_i$ open.
i.e. $U = \{(x_i)_{i\in I} \in \prod_{i\in I} X_i | i \in I, x_i \in U_i \text{ where } U_i \text{ are open in } X_i\}$. And we have $Q = \{i \in I | U_i \neq X_i\}$ finite and $I = Q \cup (I \setminus Q)$.
Consider $f^{-1}U$:

$$
\begin{aligned}
f^{-1}U &= \{y \in Y | fy \in U\} \\
&= \{y \in Y | \forall i \in I, f_i y \in U_i\} \\
&= \left\{y \in Y | \forall i \in (Q \cup I \setminus Q), y \in f_i^{-1} U_i\right\} \\
&= \bigcap_{i \in (Q \cup I \setminus Q)} f_i^{-1} U_i \\
&= \left(\bigcap_{i \in Q} f_i^{-1} U_i\right) \bigcap \left(\bigcap_{i \in I \setminus Q} \underbrace{f_i^{-1} \underbrace{U_i}_{=X_i}}_{=Y}\right) \\
&= \left(\bigcap_{i \in Q} \underbrace{f_i^{-1} U_i}_{\in \mathcal{T}_Y}\right) \bigcap Y \\
&\in \mathcal{T}_Y, \text{ because topology is closed under finite intersection.}
\end{aligned}
$$

Finally, since the choice of U is arbitrary. Hence, we proved f is continuous.

$\square$

3 L7-18 (fun torus)

Prove the torus as a finite CW-complex with one 0-cell (i.e. $|X_0| = *$), two 1-cells and one 2-cell.

Background story: For presenting finite CW complex, we need to show a sequence of $X_0, X_1, \ldots, X_n = X$ together with the attaching maps $\left\{f_\alpha : S^{i-1} \to X_{i-1}\right\}_{\alpha\in\Lambda_i}$, where λ_i is the family of cells in i^{th} attaching step. This process of attaching n-cells [2], is to attach the boundary of our n-cells S^{n-1} onto the previous presentation X_{n-1}. The function $f = \coprod_{\alpha\in\Lambda_i} f_\alpha$, with its individual components f_α indicates how do we attach those boundaries to the previous presentation X_{n-1}. The attaching process from presentation $(n-1)$ to n can be summarized as the following commute diagram:

$$\begin{array}{ccc}
\coprod_{\alpha\in\Lambda_i} S^{n-1}_\alpha & \xrightarrow{f} & X_{n-1} \\
\Big\downarrow {\scriptstyle \coprod_\alpha \iota_\alpha} & \ulcorner & \Big\downarrow \\
\coprod_{\alpha\in\Lambda_i} D^n_\alpha & \longrightarrow & X_n
\end{array}$$

Sketch of proof: Focus to our question, we have two attaching actions. What we know are:

1. $X_0 = \{*\}$ is a single point set, this is our 0-cell.
2. We have two 1-cells, which is the disjoint union of two 2-discs: $D^1 \coprod D^1$, where D^1 is an interval $D^1 = [-1, 1]$. We need to attach their boundaries $S^0 \coprod S^0\{(a, -1), (a, 1), (b, -1), (b, 1)\}$, where $S^0 = \{-1, 1\}$, to the 0-cell X_0. Note that we use a to indicate first component of the coproduct and b indicates the second component
3. We have one 2-cell, which is an 2-disc, D^2. We have to attach it to the X_1, where X_1 by inspection is the "number eight".

So we consider two such commute diagrams:

$$\begin{array}{ccc}
S^0 \coprod S^0 & \xrightarrow{f_{S^0} \coprod f_{S^0}} & X_0 \\
\Big\downarrow {\scriptstyle \imath_1 \coprod i_2} & \ulcorner & \Big\downarrow \\
D^1 \coprod D^1 & \longrightarrow & X_1
\end{array}$$

$$\begin{array}{ccc}
S^1 & \xrightarrow{f} & X_1 \\
\Big\downarrow {\scriptstyle i} & \ulcorner & \Big\downarrow \\
D^2 & \longrightarrow & X_2
\end{array}$$

---

[2] n-cell is just the n-disc D^n. $\coprod_{\alpha\in\Lambda_i} D^n_\alpha$ means there are $|\Lambda_i|$ to be attached.

More precisely, we have to come up with how to "attach" those cells, i.e. to find two attaching maps $f_{S^0} \coprod f_{S^0}$ and f, and two presentations X_1 and X_2.

The map $f_{S^0} : \{-1, 1\} \to \{*\}$ is obvious. We have $f_{S^0}(-1) = * = f_{S^0}(1)$. Note the set $S^0 \coprod S^0 = \{(a, -1), (a, 1), (b, -1), (b, 1)\}$. So

$$\forall x \in S^0 \coprod S^0, (f_{S^0} \coprod f_{S^0})(x) = *$$

The map $f : S_1 \to X_1$ is trickier. Aside of that, let's consider what is X_1 in order to get some clue about what f should be. Graphically, we know X_1 is two circles S^1 attached to a single point. S^1 here is actually $[-1, 1]/\sim = \{[1], x|x \in (-1, 1)\}$, with the equivalence relation is $1 \sim -1$. We have two such circles, and the point of attachment is where we have attached -1 and 1. Hence, we can imagine that X_1 is also a disjoint union of circles quotient out some equivalence relation. That equivalence relation will be $(a, [1]) \sim (b, [1])$ where we use a and b to indicate the $[1] \in S^1$ come from the first and second circle respectively. The disjoint union looks like $S^1 \coprod S^1 = \{(i, y)|i \in \{a, b\}, y \in S^1\}$. So we have:

$$\begin{aligned}
X_1 &= \left(S^1 \coprod S^1\right)/\sim \\
&= \{[(a, [1])], (i, x)|i \in \{a, b\}, x \in (-1, 1)\} \\
&= \{[(a, [1])]\} \cup \{(a, x)|x \in (-1, 1)\} \cup \{(b, x)|x \in (-1, 1)\} \\
&= \{[(a, [1])], (a, x)|x \in (-1, 1)\} \cup \{[(b, [1])], (b, x)|x \in (-1, 1)\} \\
&= S^1_a \cup S^1_b
\end{aligned}$$

where $[(a, [1])] = [(b, [1])] = [(a, [-1])] = [(b, [-1])]$, and we use S^1_a and S^1_b to indicate X_1 is really the union of some circles. Another thing to be noted is we can display what $D^1 \coprod D^1$ looks like:

$$D^1 \coprod D^1 = \{(i, x)|i \in \{a, b\}, x \in D^1 = [-1, 1]\}$$

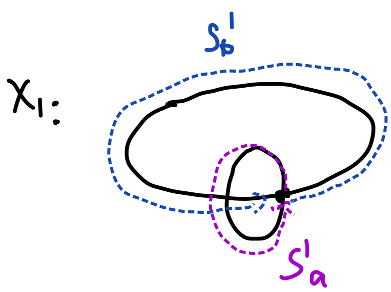
It is possible to construct a map from D^1 to S^1_a for S^1_b. Define $f_a : D^1 \to S^1_a$ and $f_b : D^1 \to S^1_b$ where they has the form

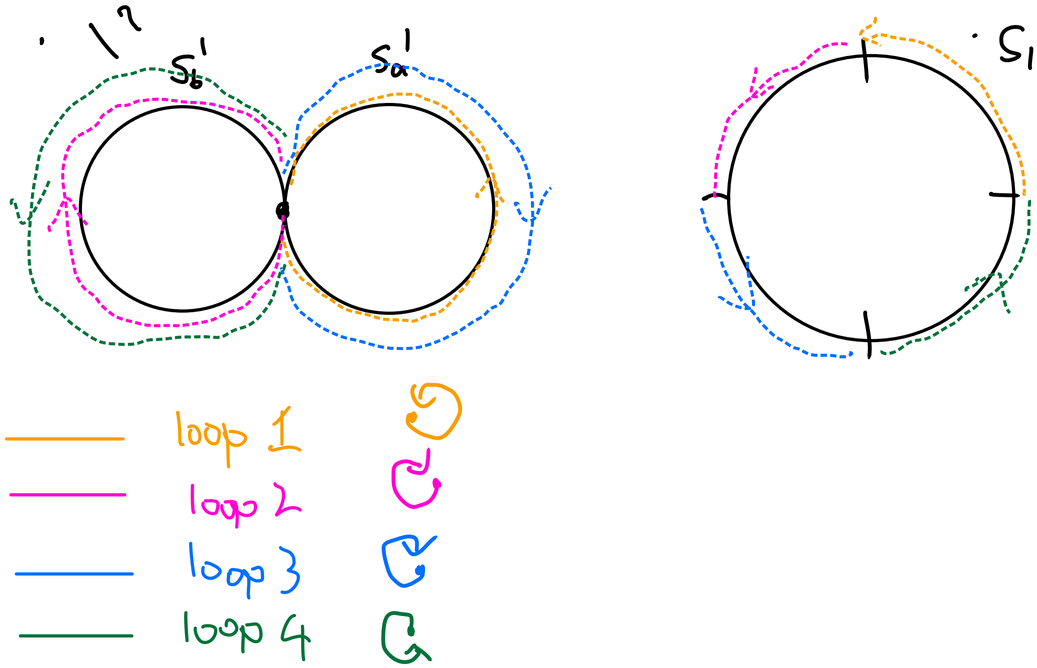
$$f_a(x) = \begin{cases} (a, x) & \text{, if } x \in (-1, 1) \\ [(a, [1])] & \text{, if } x \in \{-1, 1\} \end{cases}$$

$$f_b(x) = \begin{cases} (b, x) & \text{, if } x \in (-1, 1) \\ [(b, [1])] & \text{, if } x \in \{-1, 1\} \end{cases}$$

Observe these two functions, we can find a continuous map from $S^1 = [-1, 1]/\sim$ to D^1. Such map is $x \mapsto x, x \in (-1, 1)$ and $x \mapsto \{-1, 1\}, x = [-1] = [1]$. Then by factor through the quotient, I can find an unique continuous map between S^1 and S^1_a or S^1_b. However, I cannot imagine how to find a map from S^1 to $S^1_a \cup S^1_b$ that can let S^1 covers the entire $S^1_a \cup S^1_b$. So I decide to use some other approach by inspect how I walk around the circle using some parametric functions. First, let $(x,y) = (cos(\theta), sin(\theta))$ for some $\theta \in [0, 2\pi]$. So as θ rotates from 0 to 2π, we have walked along S^1 once. Next is my friend's contribution, she showed me how she walked around the number eight by walking along each circle twice with just walking along the S^1 once.

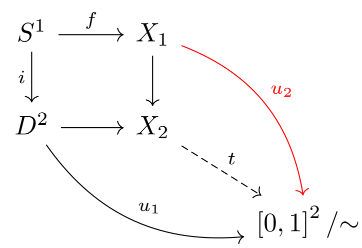
$$
\begin{aligned}
f(x,y) &= f(cos(\theta), sin(\theta)) \\
&= \begin{cases}
(a, (cos(4\theta), sin(4\theta))) & , \text{if } \theta \in \left[0, \frac{\pi}{2}\right] \\
\left(b, \left(cos\left(4\left(\theta - \frac{\pi}{2}\right)\right), sin\left(4\left(\theta - \frac{\pi}{2}\right)\right)\right)\right) & , \text{if } \theta \in \left[\frac{\pi}{2}, \pi\right] \\
(a, (cos(4(\pi - \theta)), sin(4(\pi - \theta)))) & , \text{if } \theta \in \left[\pi, \frac{3\pi}{2}\right] \\
\left(b, \left(cos\left(4\left(\frac{3\pi}{2} - \theta\right)\right), sin\left(4\left(3\frac{\pi}{2} - \theta\right)\right)\right)\right) & , \text{if } \theta \in \left[\frac{3\pi}{2}, 2\pi\right]
\end{cases} \\
&= \begin{cases}
\left(a, e^{i4\theta}\right) & , \text{if } \theta \in \left[0, \frac{\pi}{2}\right] \\
\left(b, e^{i4\left(\theta - \frac{\pi}{2}\right)}\right) & , \text{if } \theta \in \left[\frac{\pi}{2}, \pi\right] \\
\left(a, e^{i4(\pi - \theta)}\right) & , \text{if } \theta \in \left[\pi, \frac{3\pi}{2}\right] \\
\left(b, e^{i4\left(\frac{3\pi}{2} - \theta\right)}\right) & , \text{if } \theta \in \left[\frac{3\pi}{2}, 2\pi\right]
\end{cases}
\end{aligned}
$$



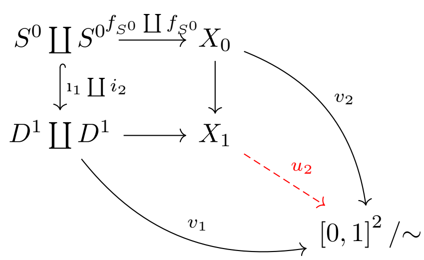


After all, we have to show our final presentation X_2 is indeed a torus. That is to find a homeomorphism between X_2, and one of the $(S^1 \times S^1)$, $([0,1]^2/\sim)$, $((S^1 \times [0,1])/\sim)$. This can be done via a pushout construction.

So the goal is to show $X_2 \cong [0,1]^2/\sim$, meaning to show a function $t : X_2 \to [0,1]^2/\sim$ is continuous, bijective and its inverse function $s : [0,1]^2/\sim \to X_2$ is also continuous. First, the continuity of t can be obtained from the pushout construction, and t is in the position of the following pushout diagram:



in order to prove such t is continuous using the theory of the pushout, we need to check for some functions u_1 and u_2, they are continuous and satisfy the contion of $u_2 \circ f = u_1 \circ i$. Note that the red arrow u_2 in above diagram contains exactly the same information as the following red-dashed arrow:



That means in order to claim that u_2 is continuous for free, we have to find another two functions v_1 and v_2 such that $v_2 \circ f_{S^0} \coprod f_{S^0} = v_1 \circ i_1 \coprod i_2$.

So to wrap up, if we can find u_1, u_2, v_1, v_2 are continuous and make the relative diagram commutes, then we can claim t is continuous for free. The only part in the process that we can use pushout construction is when we trying to find u_2. Now let's think about how to find other maps.

For $u_1 : D^2 \to [0,1]^2/\sim$. The trick is to see the 2-disc D^2 is homeomorphic to the square $[0,1]^2$. If that is established, then we can instead consider $u_1' : [0,1]^2 \to [0,1]^2/\sim$, which is exactly the quotient map. And that quotient map is exactly what we have seen in Tutorial 2. We know by definition of quotient map, it is continuous.

For $v_2 : \{*\} \to [0,1]^2/\sim$, this can be thought as attaching a single point to the four vertices of the square $[0,1]^2$ and then compose with the quotient map from $[0,1]^2 \mapsto [0,1]^2/\sim$, say $v_2 = v_2' \circ \phi$. The v_2' here is a surjective map from $\{*\}$ to four vertices on the square $[0,1]^2$, which is the map $\{*\} \mapsto \{(1,0),(0,0),(1,1),(0,1)\}$. Clearly this map is NOT injective. We know from the tutorial 2, the four vertices form an equivalence class. Hence, we can define such map v_2 to be $\{*\} \mapsto [(1,0)]$ where

$$[(1,0)] = \{(x,y) \in [0,1]^2 /\!\sim\} = \{(1,0),(0,0),(1,1),(0,1)\}$$

This established a one-to-one correspondence. The question to be asked is that is it continuous? It turns out it is. First note that the topology of $\{*\}$ is just $\{*,\varnothing\}$. Because v_2 will map all set contains that equivalence class $[(1,0)]$ back to the whole space $\{*\}$, and others to the empty set, which are both open in the single point topology.

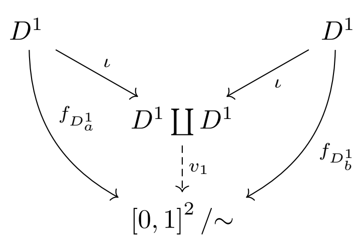
For $v_1 : D^1 \coprod D^1 \to [0,1]^2 /\!\sim$, the idea is to map the two intervals D^1 to the four edges of the square $[0,1]^2$. Previously we see that $D^1 \coprod D^1$ looks like:

$$D^1 \coprod D^1 = \{(i,x) | i \in \{a,b\}, x \in D^1 = [-1,1]\}$$

but it seems that knowing its form do not give me any inspiration. So I would incline to use the Universal Property of Disjoint Union by defining the canonical maps of injection $\iota : D^1 \to D^1 \coprod D^1$. And $f_{D^1_a} : D^1_a \to [-1,1]^2 /\!\sim$ and $f_{D^1_b} : D^1_b \to [-1,1]^2 /\!\sim$ are defined as:

$$f_{D^1_a}(x) = [(x,0)]\,, -1 \le x \le 1$$
$$f_{D^1_b}(x) = [(0,x)]\,, -1 \le x \le 1$$

where the subscript just to distinguish the two 1-cells in the disjoint union. Note here I cheat a little bit that I'm using the square $[-1,1]^2$ instead of the square $[0,1]^2$, but that's okay since we have a homeomorphism between $[0,1]$ and $[-1,1]$. Now claim these maps are well-defined and continuous. Because the function $x \mapsto (x,0)$ is continuous, and we composite this map with a quotient map, we know the composition of the continuous maps is continuous. Therefore, by Universal Property of Disjoint Union, there is a unique continuous map v_1 which let the following diagram commutes:



Hence, the continuity of v_1 is proved. But after all, eventually I found that I can just claim

$$v_1((i,x)) = \left(f_{D^1_a} \coprod f_{D^1_b}\right)((i,x)) = \begin{cases} [(x,0)]\,, & -1 \le x \le 1, i = a \\ [(0,x)]\,, & -1 \le x \le 1, i = b \end{cases}$$

Second, we have to prove the s is continuous, the following diagram shows a good indication of what we should do.

$$\begin{array}{ccc} D^2 & \longrightarrow & [0,1]^2/\sim \\ & \searrow & \downarrow{\scriptstyle s} \\ & & X_2 \end{array}$$

so from the diagram, we can get some sense that the trick is to use the Universal Property of the Quotient. If we can replace D^2 with $[0,1]^2$, we can apply Universal Property of the Quotient without hesitation. That requires us to show that the 2-disc D^2 is homeomorphic to the square $[0,1]^2$. If homeomorphism is proved, then we have the following commute diagram.

$$\begin{array}{ccc} [0,1]^2 & \xrightarrow{\phi} & [0,1]^2/\sim \\ & \searrow^{\iota_{[0,1]^2}} & \downarrow{\scriptstyle s} \\ & & X_2 \end{array}$$

From the diagram, we know the quotient map ϕ is continuous, and we know the map $\iota_{[0,1]^2} : [0,1]^2 \to X_2$ is continuous from the Exercise L7-8. Hence, we can claim s is continuous for free.

Lastly, we have to show

3.1 Preliminary Results

We will prove some results like a 2-disc D^2 is homeomorphic to a square $[0,1]^2$ in this section.

Claim: A 2-disc is defined as $D^2 = \{(x,y) \in \mathbb{R}^2 | x^2 + y^2 \leq 1\} \subseteq \mathbb{R}^2$, and a square is defined as $X = [0,1]^2 = \{(x,y) \in \mathbb{R}^2 | 0 \leq x, y \leq 1\} \subseteq \mathbb{R}^2$. Then D^2 and C are homeomorphic.

Proof. Before the proof, let's use the fact that the function $x \mapsto \frac{x+1}{2}$ is a homeomorphism between $[0,1]$ and $[-1,1]$. Thus, we can show that the 2-disc D^2 is homeomorphic to the square $Y = [-1,1]^2$ instead of $[0,1]^2$. Firstly, because they are both subset of $\mathbb{R}^2$, then their topology is the subspace topology of $\mathbb{R}^2$.

$$\mathcal{T}_Y = \left\{[-1,1]^2 \cap B_\epsilon((x,y)) | (x,y) \in \mathbb{R}^2, \epsilon > 0\right\}$$
$$\mathcal{T}_{D^2} = \left\{D^2 \cap B_\epsilon((x,y)) | (x,y) \in \mathbb{R}^2, \epsilon > 0\right\}$$

Next question is how to define a homeomorphism $f : [-1,1]^2 \to D^2$? Start by imagining what would happen if we want to stretch the boundary of a 2-disc is sitting in the square with the boundary points $(-1,-1), (-1,1), (1,-1), (1,1)$. When we say a point on the boundary of D^2 has distance 1, we really mean the quadratic norm of that point is 1. On the other hand, when we say a point on the boundary of a unit square has distance 1, the implicit metric is the maximum norm. So for example, if we have two observers, one is using the quadratic norm d_2 and the other is using the metric of the maximum norm d_∞. Note that although two observers may both observe some points that have distances "1", actually they do not agree on that distance. In fact, they have to use some machine to convert the language the other observer talk into their own language. For example, the observer who is using the d_∞ gives the other observer a coordinate, the other observer only knows the distance from their point of view. In order to acquire the coordinate from quadratic point of view, the quadratic observer has to divide the coordinate that the other observer gave in order to obtain the coordinate from their point of view. In short, the functions that maps boundary to boundary are:

$$f : \partial Y \to \partial D^2 : (x, y) \mapsto \frac{1}{d_2(x, y)}(x, y)$$
$$g : \partial D^2 \to \partial Y : (x, y) \mapsto \frac{1}{d_\infty(x, y)}(x, y)$$

where in this case the symbol ∂ indicates the boundary of each plane. Therefore, to wrap up, if we want to define a homeomorphism $f : Y \to D^2$, the candidate homeomorphism will have the form:

$$f(x, y) = \begin{cases} 0 & \text{, if } (x, y) = (0, 0) \\ \frac{d_\infty(x,y)}{d_2(x,y)}(x, y) & \text{, otherwise} \end{cases}$$

Its inverse will be $g : D^2 \to Y$

$$g(x, y) = \begin{cases} 0 & \text{, if } (x, y) = (0, 0) \\ \frac{d_2(x,y)}{d_\infty(x,y)}(x, y) & \text{, otherwise} \end{cases}$$

It is obvious that these two functions are bijections. Because when consider $f \circ g(x, y)$ and $g \circ f(x, y)$, the coefficients will just cancel each other out. So by $f \circ g(x, y) = id_Y$ and $g \circ f(x, y) = id_{D^2}$, we have they to be bijections. Next we show they are continuous. Because they are metric spaces, we just consider the usual limit arguments.

<u>Case 1</u>: Continuity at point 0. Suppose $d_\infty(x, y) \to 0$, then

$$\begin{aligned} d_2(f(x, y) - f(0, 0), f(x, y) - f(0, 0)) &= d_2(f(x, y), f(x, y)) \\ &= d_2\left(\frac{d_\infty(x, y)}{d_2(x, y)}(x, y), \frac{d_\infty(x, y)}{d_2(x, y)}(x, y)\right) \\ &= \frac{d_\infty(x, y)}{d_2(x, y)} d_2(x, y) \\ &= d_\infty(x, y) \\ &\to 0 \end{aligned}$$

Case 2: Continuity in general. For simplicity of the notation, we write $d_2(x-y, x-y) = \|x-y\|_2$. Suppose $\|((x_2,y_2)-(x_1,y_1))\|_2 \to 0$ for some $(x_1,y_1),(x_2,y_2) \in y$ distinct, then

$$\begin{aligned}\|f(x_1,y_1)-f(x_2,y_2)\|_2 &= \left\|\frac{d_\infty(x_2,y_2)}{d_2(x_2,y_2)}(x_2,y_2)-\frac{d_\infty(x_1,y_1)}{d_2(x_1,y_1)}(x_1,y_1)\right\|_2\\ &= \left\|\frac{d_\infty(x_2,y_2)}{d_2(x_2,y_2)}((x_2,y_2)-(x_1,y_1))+(x_1,y_1)\left(\frac{d_\infty(x_2,y_2)}{d_2(x_2,y_2)}-\frac{d_\infty(x_1,y_1)}{d_2(x_1,y_1)}\right)\right\|\\ &\leq \frac{d_\infty(x_2,y_2)}{d_2(x_2,y_2)}\|((x_2,y_2)-(x_1,y_1))\|_2+\left|d_2(x_1,y_1)\left(\frac{d_\infty(x_2,y_2)}{d_2(x_2,y_2)}-\frac{d_\infty(x_1,y_1)}{d_2(x_1,y_1)}\right)\right|\end{aligned}$$

by triangular inequality

$$\begin{aligned}&= \frac{d_\infty(x_2,y_2)}{d_2(x_2,y_2)}\|((x_2,y_2)-(x_1,y_1))\|_2+\left|d_\infty(x_2,y_2)\frac{d_2(x_1,y_1)}{d_2(x_2,y_2)}-d_\infty(x_1,y_1)\right|\\ &\to 0 \text{ as } \|((x_2,y_2)-(x_1,y_1))\|_2 \to 0\end{aligned}$$

In the last line, we use the fact that as $\|((x_2,y_2)-(x_1,y_1))\|_2 \to 0$, the fraction $\frac{d_2(x_1,y_1)}{d_2(x_2,y_2)} \to 1$.

By symmetry, we can do the exactly the same thing to function g. Hence, we proved function $f: Y \to D^2$ is a homeomorphism. And thus Y and D^2 are homeomorphic.

□

3.2 Display the presentation of the finite CW complex of a torus

In this section, we will display the attaching map of attaching two 1-cells, $f_{S^0} \coprod f_{S^0} : S^0 \coprod S^0 \to X_0$. And the attaching map $f: S^1 \to X_1$.

We start from $X_0 = \{*\}$. The map $f_{S^0} : \{-1,1\} \to \{*\}$ is obvious. We have $f_{S^0}(-1) = * = f_{S^0}(1)$. Note the set $S^0 \coprod S^0 = \{(a,-1),(a,1),(b,-1),(b,1)\}$. So

$$\forall x \in S^0 \coprod S^0, (f_{S^0} \coprod f_{S^0})(x) = *$$

Then we obtain X_1 from some pushout construction:

$$\begin{aligned}X_1 &= \{[(a,[1])]\} \cup \{(a,x)|x \in (-1,1)\} \cup \{(b,x)|x \in (-1,1)\}\\ &= \{[(a,[1])],(a,x)|x \in (-1,1)\} \cup \{[(b,[1])],(b,x)|x \in (-1,1)\}\\ &= S^1_a \cup S^1_b\end{aligned}$$

where $[(a,[1])] = [(b,[1])] = [(a,[-1])] = [(b,[-1])]$, and we use S^1_a and S^1_b to indicate X_1 is really the union of some circles.

Eventually, I decide to abandon using any universal properties and instead to find the attaching map f by considering how do I walk around the circle while I'm walking around the number eight. I will use the usual definition of the circle $S^1 = \{(x,y)|x^2+y^2=1\}$. We consider (x,y) using some parametric function of the angle θ to indicate how I walk around the circle. The map is following:

$$
\begin{aligned}
f(x,y) &= f(cos(\theta), sin(\theta)) \\
&= \begin{cases}
(a, (cos(4\theta), sin(4\theta))) & , \text{if } \theta \in \left[0, \frac{\pi}{2}\right] \\
\left(b, \left(cos\left(4\left(\theta - \frac{\pi}{2}\right)\right), sin\left(4\left(\theta - \frac{\pi}{2}\right)\right)\right)\right) & , \text{if } \theta \in \left[\frac{\pi}{2}, \pi\right] \\
(a, (cos(4(\pi - \theta)), sin(4(\pi - \theta)))) & , \text{if } \theta \in \left[\pi, \frac{3\pi}{2}\right] \\
\left(b, \left(cos\left(4\left(\frac{3\pi}{2} - \theta\right)\right), sin\left(4\left(3\frac{\pi}{2} - \theta\right)\right)\right)\right) & , \text{if } \theta \in \left[\frac{3\pi}{2}, 2\pi\right]
\end{cases} \\
&= \begin{cases}
\left(a, e^{i4\theta}\right) & , \text{if } \theta \in \left[0, \frac{\pi}{2}\right] \\
\left(b, e^{i4\left(\theta - \frac{\pi}{2}\right)}\right) & , \text{if } \theta \in \left[\frac{\pi}{2}, \pi\right] \\
\left(a, e^{i4(\pi - \theta)}\right) & , \text{if } \theta \in \left[\pi, \frac{3\pi}{2}\right] \\
\left(b, e^{i4\left(\frac{3\pi}{2} - \theta\right)}\right) & , \text{if } \theta \in \left[\frac{3\pi}{2}, 2\pi\right]
\end{cases}
\end{aligned}
$$

3.3 Homeomorphism between torus and X_2

Use $[0,1]^2/\sim$ to denote the torus, and $t : X_2 \to [0,1]^2/\sim$ be the function that maps X_2 to the torus. In addition, denote $s : [0,1]^2/\sim \to X_2$ be the candidate inverse function of t. We want to show

1. t is continuous
2. s is continuous
3. t is bijective. i.e. $t \circ s = id_{X_2}$ and $s \circ t = id_{[0,1]^2/\sim}$

Claim: t is continuous

Proof. From the introduction part, we see the key to show t is continuous is to use the Universal Property of Pushout twice. That is to find u_1, u_2, v_1, v_2 that make those diagrams commute.

Because D^2 is homeomorphic to $[0,1]^2$, $u_1 : D^2 \to [0,1]^2/\sim$ is the quotient map $\phi : [0,1]^2 \to [0,1]^2/\sim$, which is defined as $\phi(x) = [x]$.

u_2 can be proved to be continuous using another time of Universal Property of Pushout via constructing the continuous fucntions v_1, v_2.

For $v_2 : \{*\} \to [0,1]^2/\sim$, it is the map of compositing the surjective map $v_2' : \{*\} \mapsto \{(1,0),(0,0),(1,1),(0,1)\}$ with a quotient map ϕ that maps those four points to a single equivalence class. So $v_2 = v_2' \circ \phi$. And it has the form

$$\{*\} \mapsto [(1,0)]$$

where

$$[(1,0)] = \{(x,y) \in [0,1]^2/\sim\} = \{(1,0),(0,0),(1,1),(0,1)\}$$

For v_1, we can define v_1 to be

$$v_1\left((i,x)\right) = \left(f_{D_a^1} \coprod f_{D_b^1}\right)\left((i,x)\right) = \begin{cases} \left[(x,0)\right], & -1 \leq x \leq 1, i = a \\ \left[(0,x)\right], & -1 \leq x \leq 1, i = b \end{cases}$$

with its component functions $f_{D_a^1} : D_a^1 \to \left[-1,1\right]^2/\!\sim$ and $f_{D_b^1} : D_b^1 \to \left[-1,1\right]^2/\!\sim$ are defined as:

$$\begin{aligned} f_{D_a^1}(x) &= \left[(x,0)\right], -1 \leq x \leq 1 \\ f_{D_b^1}(x) &= \left[(0,x)\right], -1 \leq x \leq 1 \end{aligned}$$

It's easy to see that the component functions are well-defined and continuous. Because the function $x \mapsto (x,0)$ is continuous, and we composite this map with a quotient map, we know the composition of the continuous maps is continuous. Lastly, by the Universal Property of Disjoint Union, we can claim that v_1 is unique and continuous.

Now, clearly we have $v_2 \circ (f_{S^0} \coprod f_{S^0}) = v_1 \circ (\imath_1 \coprod i_2)$. Because when we depart from $S^0 \coprod S^0 = \{(a,-1),(a,1),(b,-1),(b,1)\}$, there is only one destination we can arrive at $[-1,1]^2/\!\sim$, which is

$$\left[(1,0)\right] = \{(x,y) \in \left[0,1\right]^2/\!\sim\} = \{(1,0),(0,0),(1,1),(0,1)\}$$

Thus, we can claim that u_2 is continuous by the Universal Property of Pushout.

Now we have u_1, u_2 both continuous, we want to know whether $u_1 \circ i = u_2 \circ f$. It's hard to show they commute by put in some specific points. But we know u_2, u_1 are all unique, and f and i both have some sin and cos, so I will just claim that they commute from some graph intuition. Hence, we proved that t is continuous by the Universal Property of Pushout.

□

Claim: s is continuous

Proof. We proved in previous section that D^2 is homeomorphic to $[-1,1]^2/\!\sim$, and by transitivity of homeomorphism, D^2 is homeomorphic to $[0,1]^2/\!\sim$. Therefore, instead of considering the map $D^2 \to [0,1]^2/\!\sim$, we can consider the map $[0,1]^2 \to [0,1]^2/\!\sim$. Note that such map will be the quotient map ϕ. By the pushout construction, we know the map $D^2 to X_2$ is continuous. Thus, by the Universal Property of Quotient, there is a unique continuous map s making the following diagram commutes.

$$\begin{array}{ccc} \left[0,1\right]^2 & \xrightarrow{\phi} & \left[0,1\right]^2/\!\sim \\ & \searrow^{\iota} & \downarrow{\scriptstyle s} \\ & & X_2 \end{array}$$

□

Finally, we know both t and s are unique continuous functions that maps each of their domain to their codomain, so the only possible solution of that is they must be inverse to each other. Hence, the bijectivity is proved.

Overall, we proved t is a homeomorphism to X_2 and $[0,1]^2/\sim$. Therefore, this presentaion of finite CW complex present us a torus with the attaching map.

4 L7-21 (circle represents periodic functions)

Given $P > 0$, let $\sim$ be the equivalence relation on $\mathbb{R}$ generated by $x \sim x + P$ for all $x \in \mathbb{R}$. Prove that $\mathbb{R}/{\sim} \cong S^1$ (i.e. homeomorphic to S^1 and hence that there is a bijection for any space Y such that

$$\text{Cts}\left(S^1, Y\right) \cong \{f : \mathbb{R} \to Y | f \text{ is continuous and } \forall x \in \mathbb{R}, f(x) = f(x + P)\})$$

The circle is the space that represents periodic continuous functions.

Background story: In this section, we have a 1-sphere $S^1 = \left\{(x, y) | x^2 + y^2 = 1\right\} \subseteq \mathbb{R}^2$. We have two equivalence relations. To make these two equivalence relations distinct, let's name these equivalence relations. Denote the first equivalence relation of $x \sim x + P$ for some $P > 0$, R_1. i.e.

$$R_1 = \{(x, x + P) | x \in \mathbb{R}\} \subseteq \mathbb{R} \times \mathbb{R} \text{ for some } P > 0.$$

One can see that the equivalence class for an arbitrary $x \in \mathbb{R}$ will look like:

$$[x] = \{\dots, x - 2P, x - P, x, x + P, x + 2P, \dots\} = \{x + nP | n \in \mathbb{Z}\}$$

We have another equivalence relation generated by R_1. Call it R_2
Sketch of proof: For questions like proving the something is homeomorphic to a quotient space, I will have the Universal Property of the quotient in my mind.

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{\phi} & \mathbb{R}/{\sim} \\ & \searrow^{f \circ \phi} & \downarrow f \\ & & S^1 \end{array}$$

Where in this case, we have the quotient map $\phi : \mathbb{R} \to \mathbb{R}/{\sim}$ which has the form $\phi(x) = [x]$ for $x \in \mathbb{R}$. And $f : \mathbb{R}/{\sim} \to S^1$ is the function of interest. We want to show f is a homeomorphism. Then the steps of solving will become following:

1. Finding a easier version of function $f : \mathbb{R}/{\sim} \to S^1$, which is $f \circ \phi : \mathbb{R} \to S^1$. And this function $f \circ \phi$ needs to be (1) Well-defined, (2) satisfies the equivalence relation that:

   $$\forall x \in \mathbb{R}, (f \circ \phi)(x) = (f \circ \phi)(x + P) \text{ whenever } x \sim x + P$$

   This property will allow us to make connection between the function f and the function $f \circ \phi$ in the following way:

   $$(f \circ \phi)(x) = f([x])$$

   Therefore, if $f \circ \phi$ is well-define, then f is also well-defined. By the Universal Property of the Quotient, such f is continuous. Hence, the continuity of the candidate homeomorphism is proved.

2. We find the inverse of function $f : \mathbb{R}/\sim \to S^1$. Call it $g : S^1 \to \mathbb{R}/\sim$. We need to check (1) g is bijective by checking :

$$f \circ g = id_{\mathbb{R}/\sim} \text{ and } g \circ f = id_{S^1}$$

And (2): check g is continuous.

In part 2, define the function:

$$\Phi : \text{Cts}\left(S^1, Y\right) \to \{\mathbb{R} \to Y | f \text{ is continuous and } \forall x \in \mathbb{R} f(x) = f(x + P)\}$$

which is given by $\Psi(F) = F \circ \psi$ for $F \in \text{Cts}\left(S^1, Y\right)$. So we may want to show the following diagram commutes:

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{\psi} & S^1 \\ & \overset{f}{\searrow} & \downarrow F \\ & & Y \end{array}$$

We have to show such Ψ is bijective. We note that the function F maps from S^1 to Y. However, since we have already showed there is a homeomorphism between $\mathbb{R}/\sim$ and S^1, so we expect the following diagram commutes:

$$\begin{array}{ccc} S^1 & \xrightarrow{f} & \mathbb{R}/\sim \\ & \overset{F}{\searrow} & \\ & & Y \end{array}$$

therefore, we are allowed to define a new family of continuous functions $G \in \text{Cts}(\mathbb{R}/\sim, Y)$, and instead consider the following diagram:

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{\phi} & \mathbb{R}/\sim \\ & \overset{f}{\searrow} & \downarrow G \\ & & Y \end{array}$$

where Φ is defined as $\Phi(G) = G \circ \phi$ for $G \in \text{Cts}(\mathbb{R}/\sim, Y)$ and ϕ be the quotient map. Remember the goal is to show Ψ is bijective. This can be achieved by showing the function Φ is bijective. So we show Φ is injective and surjective.

4.1 To show $\mathbb{R}/\sim \cong S^1$

Proof. Inspect that $(f \circ \phi)(t) = \left(cos\left(\frac{2\pi}{P}t\right), sin\left(\frac{2\pi}{P}t\right)\right)$.
Check that this is well-defined. Consider an arbitrary $t \in \mathbb{R}$.

$$\begin{aligned}(f \circ \phi)(t+P) &= \left(cos\left(\frac{2\pi}{P}(t+P)\right), sin\left(\frac{2\pi}{P}(t+P)\right)\right)\\ &= \left(cos\left(\frac{2\pi}{P}t+2\pi\right), sin\left(\frac{2\pi}{P}t+2\pi\right)\right)\\ &= \left(cos\left(\frac{2\pi}{P}t\right), sin\left(\frac{2\pi}{P}t\right)\right)\\ &= (f \circ \phi)(t)\end{aligned}$$

Because the choice of $t \in \mathbb{R}$ is arbitrary, thus the function is well-defined. Hence, the function $f : \mathbb{R}/\sim \to S^1$ is defined as $(f \circ \phi)(t) = f([t])$ for some $t \in \mathbb{R}$. Here, we use the fact that trigonometry functions are continuous without proving. Hence, by universal property of the quotient, the function f is continuous.

Next, we define a candidate inverse function $g : S^1 \to \mathbb{R}/\sim$. For each $(x, y) \in S^1$, we can uniquely define $(x, y) = \left(cos\left(\frac{2\pi}{P}t\right), sin\left(\frac{2\pi}{P}t\right)\right)$ for $t \in [0, P)$. (i.e. we have walked around the circle for nearly one period). As the clock hit the point $t + P$, we have $(x(t), y(t)) = (x(t+P), y(t+P))$. But $t \sim t + P$!. Hence we are allowed to define $g(x(t), y(t)) = [t]$ for any $t \in \mathbb{R}$.

Let $t \in \mathbb{R}$, implies $[t] \in \mathbb{R}/\sim$. Consider $g \circ f$:

$$\begin{aligned}g \circ f([t]) &= g(f([t]))\\ &= g\left(cos\left(\frac{2\pi}{P}t\right), sin\left(\frac{2\pi}{P}t\right)\right)\\ &= [t]\end{aligned}$$

Let $(x, y) \in S^1$ which is defined as $(x, y) = \left(cos\left(\frac{2\pi}{P}t\right), sin\left(\frac{2\pi}{P}t\right)\right)$. Consider $f \circ g$:

$$\begin{aligned}f \circ g(x, y) &= f(g\left(cos\left(\frac{2\pi}{P}t\right), sin\left(\frac{2\pi}{P}t\right)\right))\\ &= f([t])\\ &= \left(cos\left(\frac{2\pi}{P}t\right), sin\left(\frac{2\pi}{P}t\right)\right)\end{aligned}$$

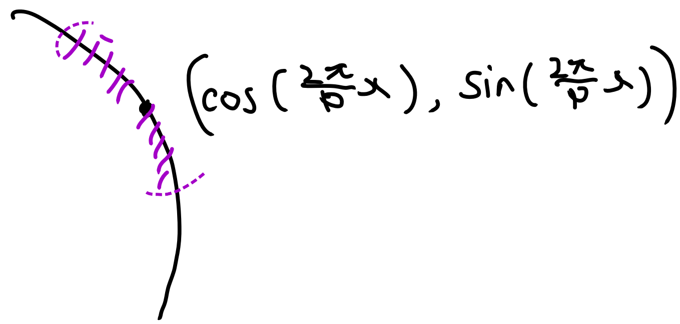
It shows that f & g are bijective.

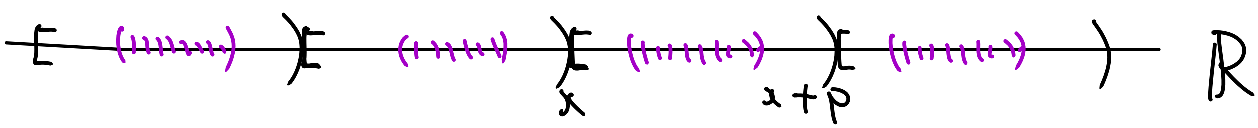
Next, to show g is continuous, consider the preimage of an arbitrary open set $U \subseteq \mathbb{R}/\sim$. Let's use the arc length metric d_a on S^1.

$$g^{-1}U = \{[t] \in \mathbb{R}/\sim | g([t]) \in U\} = fU = B_{\epsilon}^{da}\left(\frac{2\pi t}{P}\right)$$

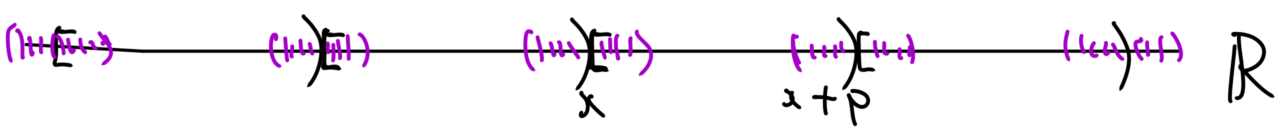
for some $\frac{2\pi t}{P} \in [0, 2\pi)$. The last equality hold is because if take any open balls on the real line mod out some equivalence relation, you will always has the corresponding open balls on the arc. [See picture]

$\square$





Infinite copies of



You see? At the boundary we can still have those open sets

4.2 To show the second homeomorphism

Given two continuous functions $G \in \mathrm{Cts}(\mathbb{R}/\sim, Y)$ and $f \in \mathrm{Cts}(\mathbb{R}, Y)$. We construct a map $\phi : \mathbb{R} \to \mathbb{R}/\sim$ and show the map ϕ is bijective.

Proof. Given $G_1, G_2 \in \mathrm{Cts}(\mathbb{R}/\sim, Y)$ be arbitrary. Suppose $\phi(G_1) = \phi(G_2)$.

$$\begin{aligned}
&\phi(G_1) = \phi(G_2)\\
&\implies (G_1 \circ \phi) = (G_2 \circ \phi)\\
&\implies (G_1 \circ \phi)(y) = (G_2 \circ \phi)(y), \forall y \in \mathbb{R}\\
&\implies G_1([y]) = G_2([y]), \forall [y] \in \mathbb{R}/\sim\\
&\implies G_1 = G_2
\end{aligned}$$

Hence, it proves that ϕ is injective. Next, we prove surjective. Let $f \in \mathrm{Cts}(\mathbb{R}, Y)$ be arbitrary, we want to show

$$\exists G \in \mathrm{Cts}(\mathbb{R}/\sim, Y) \text{ such that } \Phi(G) = f$$

So the task becomes showing G satisfies the definition of $f = \Phi(G) = G \circ \phi$ is continuous. So we want to show

$$\forall U \subseteq Y, \left(U \in \mathcal{T}_Y \implies G^{-1}(U) \in \mathcal{T}_{\mathbb{R}/\sim}\right)$$

$$\begin{aligned}
G^{-1}(U) &= \{[t] \in \mathbb{R}/\sim | G([t]) \in U\}\\
&= \{[t] \in \mathbb{R}/\sim | G(\phi(t)) \in U\}\\
&= \{[t] \in \mathbb{R}/\sim | f(t) \in U\}\\
&= \left\{[t] \in \mathbb{R}/\sim | t \in f^{-1}(U)\right\}\\
&\in \mathcal{T}_{\mathbb{R}/\sim}, \text{ by definition of quotient topology}
\end{aligned}$$

Hence, we showed that Φ is surjective. And hence we have Φ bijective.

$\square$