MAST30026: Metric and Hilbert Spaces: Assignment 1 Tutor: Daniel Murfet Tutorial time: Wednesday, 10am

Student: Brett Eskrigge

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Two Circles

In the following exercise, we make use of:

Definition. If (X, d_X), (Y, d_Y) are metric spaces, a function $f : X \to Y$ is distance preserving if

$$d_Y(f(x_1), f(x_2)) = d_X(x_1, x_2) \quad \forall x_1, x_2 \in X$$

A distance preserving function which is bijective is called an isometry.

Exercise L6-2

Prove that (S^1, d_a), (S^1, d_2) are not isometric (that is, no isometry exists between them), but that $\mathcal{T}_{d_a} = \mathcal{T}_{d_2}$, i.e. in the associated topologies on S^1 the same sets are declared open.

Proof (not isometric). We notice that over S_1, the maximal possible distance between two points under d_a is π, whereas the maximum distance between two points under d_2 is 2 (which is *less* than π). So since there is no map that takes $x, y \in S^1$ to make $d_2(x, y) = \pi$, we can conclude that no isometry exists between (S^1, d_a) and (S^1, d_2), so they are not isometric. $\square$

Proof (topologies are equal). To show $\mathcal{T}_{d_a} = \mathcal{T}_{d_2}$, we first show $\mathcal{T}_{d_a} \subseteq \mathcal{T}_{d_2}$, then $\mathcal{T}_{d_2} \subseteq \mathcal{T}_{d_a}$.
$\mathcal{T}_{d_a} \subseteq \mathcal{T}_{d_2}$:
If $U \in \mathcal{T}_{d_a}$, then $U \subseteq S^1$ is such that $\forall x \in U\ \exists \epsilon > 0\ B^{d_a}_\epsilon(x) \subseteq U$. Now, if we wish to find the chord length that $B\epsilon^{d_a}(x)$ forms around the unit circle, we use the Pythagorean theorem to find that it is equal to

$$2 \sin\left(\frac{\epsilon}{2}\right)$$

We take $\epsilon' = 2\sin\left(\frac{\epsilon}{2}\right)$, to see that $B^{d_2}_{\epsilon'}(x) \subseteq B^{d_a}_\epsilon(x)$. Hence $U \in \mathcal{T}_{d_2}$.

$\underline{\mathcal{T}_{d_2} \subseteq \mathcal{T}_{d_a}}$:
If $U \in \mathcal{T}_{d_2}$, then $U \subseteq S^1$ is such that $\forall x \in U\ \exists \epsilon > 0\ B_\epsilon^{d_2}(x) \subseteq U$. Now, if we wish to find the "angle" that $B_\epsilon^{d_2}(x)$ forms around the unit circle, we use the cosine rule ($a^2 = b^2 + c^2 - 2bc\cos\theta$) with $b = c = 1$ (as we are on the unit circle) and $a = \epsilon$. So,

$$\begin{aligned} \epsilon^2 &= 2 - 2\cos\theta \\ \implies 1 - \frac{\epsilon^2}{2} &= \cos\theta \\ \implies \theta &= \arccos\left(1 - \frac{\epsilon^2}{2}\right) \end{aligned}$$

We take $\epsilon' = \arccos\left(1 - \frac{\epsilon^2}{2}\right)$, to see that $B_{\epsilon'}^{d_a}(x) \subseteq B_\epsilon^{d_2}(x)$. Hence $U \in \mathcal{T}_{d_a}$. $\square$

Sierpiński

In the following exercise, we make use of:

Definition. A topological space is a pair $(X, \mathcal{T})$ where X is a set and $\mathcal{T}$ is a set of subsets of X, such that

(T1) $\emptyset, X$ both belong to $\mathcal{T}$

(T2) if $U, V \in \mathcal{T}$ then $U \cap V \in \mathcal{T}$

(T3) if $\{V_i\}_{i \in I}$ is any indexed set with $V_i \in \mathcal{T}$ for all $i \in I$, then $\bigcup_{i \in I} V_i \in \mathcal{T}$.

We call such a set $\mathcal{T}$ a topology on X and say that the sets $V \in \mathcal{T}$ are open in the topology. A set $C \subseteq X$ is closed in the topology if there exists $U \in \mathcal{T}$ with $C = \overline{X \setminus U}$.

Claim. Every singleton is closed in a metrisable space.

Proof. Let us denote our metrisable space with $(X, \mathcal{T}_d)$. Let $\{*\} \subseteq X$ be a singleton in X. To prove our claim, we need only show that $X \setminus \{*\}$ is open. So we need to prove that $\forall x \in X \setminus \{*\}\ \exists \epsilon > 0\ B_\epsilon(x) \subseteq X \setminus \{*\}$.
Let our $x \in X \setminus \{*\}$ be given (as if $X \setminus \{*\}$ were empty, our statement would be vacuously true). This means that $d(*, y) > 0$.
Set $\epsilon = d(*, y)/2$. Clearly, $B_\epsilon(x) \subseteq X \setminus \{*\}$, which completes the proof. □

Exercise L6-3

Prove that $X = \{0, 1\}$ with $\mathcal{T} = \{\emptyset, X, \{1\}\}$ is a topological space. This is called the Sierpiński space and is usually denoted Σ. Prove that Σ is not metrisable.

Proof. (T1) is clear, (T2) Intersections can only be one of $\emptyset, \{1\}, X$, so it is clear, (T3) Only possible unions are $\emptyset, \{1\}, X$, so it is also clear. Hence, Σ is a topological space.

To show that Σ is not metrisable, we need only consider the above claim (since $\{1\}$, a singleton, is open in Σ). □

Fake Interval

In the following exercise, we make use of:

Definition. Let $(X, \mathcal{T})$, $(Y, \mathcal{S})$ be topological spaces. A continuous map $f : (X, \mathcal{T}) \to (Y, \mathcal{S})$ is a function $f : X \to Y$ with the property that

$$\forall V \subseteq Y \left(V \in \mathcal{S} \implies f^{-1}(V) \in \mathcal{T} \right)$$

Exercise L6-11

Consider the topological space $(X, \mathcal{T})$ with $X = [0, 1]$ and $\mathcal{T} = \left\{\emptyset, X, \left[0, \frac{1}{2}\right], \left(\frac{1}{2}, 1\right]\right\}$. Classify all the continuous function $X \to \mathbb{R}$.

Claim. A function is continuous $X \to \mathbb{R}$ if and only if it is of the form,

$$f(x) = \begin{cases} a & x \in \left[0, \frac{1}{2}\right] \\ b & x \in \left(\frac{1}{2}, 1\right] \end{cases}$$

for some $a, b \in \mathbb{R}$.

Proof. $(\Rightarrow)$ Clearly, any continuous function $X \to \mathbb{R}$ must be a hybrid function (due to the discreteness of $\mathcal{T}$). So let us suppose (for a contradiction), that we have some other hybrid function $g \in \text{Cts}(X, R)$ with steps in *different* domains.

That is, for $i \in I$ we have $a_i \in \mathbb{R}$ and $A_i \subseteq X$ forming a partition of X, such that $A_i \notin \left\{\left[0, \frac{1}{2}\right], \left(\frac{1}{2}, 1\right]\right\}$ and,

$$g(x) = \begin{cases} a_i & x \in A_i \end{cases}$$

[note that $|I| \geq 2$, as $|I| = 1$ means that $f = g$, with $a = b$]

And since g is continuous, any $U \subseteq \mathbb{R}$ open $\implies g^{-1}(U) \in \mathcal{T}$. So let us take $U \subseteq \mathbb{R}$ to be such that $a_i \in U$ and $a_j \notin U$ for all $j \neq i$, for some $i \in I$. This implies that $g^{-1}(U) = A_i \in \mathcal{T}$. But this contradicts the definition of $\mathcal{T}$.

Hence, $g \notin \text{Cts}(X, \mathbb{R})$. So we conclude that if a function is in $\text{Cts}(X, \mathbb{R})$, it must be of the form listed above.

$(\Leftarrow)$ Clearly the preimage of f will be one of $\emptyset, X, \left[0, \frac{1}{2}\right], \left(\frac{1}{2}, 1\right]$ (which are precisely the elements of $\mathcal{T}$) for any subset of $\mathbb{R}$. And in particular, the open subsets of $\mathbb{R}$.

□

Product

In the following exercise, we make use of:

Lemma (L7-1)**.** Let X be a set and $\mathcal{B}$ a collection of subsets of X satisfying

(B1) For each $x \in X$ there exists $B \in \mathcal{B}$ with $x \in B$

(B2) Given $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$ there exists $B_3 \in \mathcal{B}$ with $x \in B_3 \subseteq B_1 \cap B_2$

Then there is a unique topology $\mathcal{T}$ on X for which $\mathcal{B}$ is a basis. We call $\mathcal{T}$ the topology generated by $\mathcal{B}$.

Definition. Let $\{X_i\}_{i \in I}$ be an indexed family of topological spaces. The product space $\prod_{i \in I} X$ is the usual product set with the topology generated by the basis consisting of sets

$$\prod_{i \in I} U_i = \left\{ (x_i)_{i \in I} \prod_{i \in I} X_i \;\middle|\; x_i \in U_i \text{ for all } i \right\}$$

where each $U_i \subseteq X_i$ is open and the set $\{i \in I \mid U_i \neq X_i\}$ is finite.
(i.e. something like $\ldots \times X_{-2} \times X_{-1} \times U_0 \times U_1 \times \ldots \times U_k \times X_{k+1} \times \ldots$ if $I = \mathbb{Z}$)

Exercise L7-2

Prove that $\prod_i U_i$ as defined above satisfy (B1), (B2), so that the topology on $\prod_{i \in I} X_i$ is well-defined.

Proof. (B1) We may take $U_i = X_i$ for every $i \in I$. This is a valid thing to do since now $\{i \in I \mid U_i = X_i\} = \emptyset$, which is certainly finite.

This means that $\prod_{i \in I} X_i \in \mathcal{B}$, so we can pick any $(x_i)_{i \in I} \in \prod_i X_i$, and it will lie within at least one of the basis elements (namely, $\prod_i X_i$ itself). So (B1) is satisfied.

(B2) We are concerned with subsets of the form $\prod_{i \in I} U_i$ and $\prod_{i \in I} V_i$, where only finitely many of the U_i's and V_i's are not equal to X_i.

Let,

$$\begin{aligned}
&(x_i)_{i \in I} \in \left(\prod_{i \in I} U_i \right) \cap \left(\prod_{i \in I} V_i \right) \\
\iff &(x_i)_{i \in I} \in \prod_{i \in I} U_i \text{ and } (x_i)_{i \in I} \in \prod_{i \in I} V_i \\
\iff &x_i \in U_i \quad \forall i \in I \text{ and } x_i \in V_i \quad \forall i \in I \\
\iff &x_i \in U_i \cap V_i \quad \forall i \in I \\
\iff &(x_i)_{i \in I} \in \prod_{i \in I} (U_i \cap V_i)
\end{aligned}$$

Hence,

$$\left(\prod_{i\in I} U_i\right) \cap \left(\prod_{i\in I} V_i\right) = \prod_{i\in I} (U_i \cap V_i)$$

Now, we claim that $\prod_{i\in I}(U_i \cap V_i)$ is in the basis. To show this, we need only show that $\{i \in I \mid U_i \cap V_i \neq X_i\}$ is finite (since $U_i \cap V_i$ is clearly open).

So $U_i \cap V_i \neq X_i$ will hold if and only if at least one of U_i or V_i are not X_i. This means that

$$\{i \in I \mid U_i \cap V_i \neq X_i\} = \{i \in I \mid U_i \neq X_i\} \cup \{i \in I \mid V_i \neq X_i\}$$

Now both $\{i \in I \mid U_i \neq X_i\}$ and $\{i \in I \mid V_i \neq X_i\}$ are finite, so $\{i \in I \mid U_i \cap V_i \neq X_i\}$ must be finite as well.

Hence, $\prod_{i\in I}(U_i \cap V_i)$ is a basis element.

Now, we may pick $x \in \left(\prod_{i\in I} U_i\right) \cap \left(\prod_{i\in I} V_i\right)$, and we see that if we take $B = \prod_{i\in I}(U_i \cap V_i)$ as a basis element, that $x \in B \subseteq (\prod_i U_i) \cap (\prod_i V_i)$. So (B2) is satisfied.

So by Lemma L7-1, the basis of the product topology is well defined, and it uniquely defines a topology of the product. □

R-Omega

In the following exercises, we make use of:

Definition. A metric space is a pair (X, d) consisting of a set X and a function

$$d : X \times X \to \mathbb{R}$$

satisfying the axioms:

(M1) $d(x,y) \geq 0 \quad \forall x, y \in X$ (non-negativity)

(M2) $d(x,y) = 0 \iff x = y \quad \forall x, y \in X$ (separation)

(M3) $d(x,y) = d(y,x) \quad \forall x, y \in X$ (symmetry)

(M4) $d(x,y) + d(y,z) \geq d(x,z) \quad \forall x, y, z \in X$ (triangle inequality)

Exercise (L6-10)**.** The metrics d_1, d_2 and d_∞ (on $\mathbb{R}^2$) are all Lipschitz equivalent, so

$$(\mathbb{R}^2, \mathcal{T}_{d_1}) = (\mathbb{R}^2, \mathcal{T}_{d_2}) = (\mathbb{R}^2, \mathcal{T}_{d_\infty})$$

Exercise L7-4 (i)

Prove $\mathbb{R}^n$ (with the metric topology) is equal as a topological space to the product of n copies of $\mathbb{R}$, in the above sense.

Proof. Let us denote the topology of $\mathbb{R}^n$ (with the metric topology) by $\mathcal{T}_1$, and the product topology of n copies of $\mathbb{R}$ by $\mathcal{T}_2$.
We proved in an Exercise L6-10 that $\mathbb{R}^2$ has equivalent induced topologies under the metrics d_1, d_2, or d_∞. So we may choose any of these metrics for $\mathbb{R}^2$ as we please. This can be extended to $\mathbb{R}^n$, so we will choose d_2 as our metric for $\mathbb{R}^n$.

$\mathcal{T}_1 \subseteq \mathcal{T}_2$:
Let us take $x \in \mathbb{R}^n$ and some $U_1 \in \mathcal{T}_1$ such that $x \in U_1$. Since U_1 is open, we can find a ball $B_\epsilon^{d_2}(x) \subseteq U_1$ (for some $\epsilon > 0$), which contains x. This ball contains the open box $U_2 \in \mathcal{T}_2$ where $U_2 := (x - \delta, x + \delta)^n$ (that is, the product of n copies of $(x - \delta, x + \delta)$), where $\delta \leq n^{-1/2}\epsilon$. This open box clearly contains x.
This means that $x \in U_2 \subseteq B_\epsilon^{d_2}(x) \subseteq U_1$. So we have proved that the metric topology is contained within the product topology.

$\mathcal{T}_2 \subseteq \mathcal{T}_1$:
Again, we take $x \in \mathbb{R}^n$, and some $U_2 \in \mathcal{T}_2$ such that $x \in U_2$. Since U_2 is generated by unions of the products of open sets in $\mathbb{R}$ (by definition of the product topology), we can find a box $(x - \delta, x + \delta)^n \subseteq U_2$ (for some $\delta > 0$), which contains x. This box contains the ball $U_1 \in \mathcal{T}_1$ where $U_1 := B_\epsilon^{d_2}(x)$, where $\epsilon \leq \delta/2$. This ball clearly contains x.
This means that $x \in U_1 \subseteq (x - \delta, x + \delta)^n \subseteq U_2$. So we have proved that the product topology is contained within the metric topology. $\square$

Exercise L7-4 (ii)

Is the space $\mathbb{R}^\omega := \prod_{n\in\mathbb{N}} \mathbb{R}$ metrisable? Prove it, either way.

Claim. $\mathbb{R}^\omega$ is metrisable, with metric $d(x,y) = \sup_{n\in\mathbb{N}} \left(\frac{\min\{1,|x_n-y_n|\}}{n}\right)$

Proof. The following proof makes reference to this website, in inspiration for a suitable metric.

Before we proceed with our proof, we must of course prove that d is in fact a metric for $\mathbb{R}^\omega$:

d is a metric:

(M1) Clear, since neither 1 or |-| are negative.

(M2) Clear, since $d(x,y) = 0$ iff $|x_n - y_n| = 0$ for all $n \in \mathbb{N}$ iff $x = y$.

(M3) Clear, since $|x_n - y_n|$ is symmetric about x_n and y_n for all $n \in \mathbb{N}$.

(M4) Suppose we are given $x, y, z \in \mathbb{R}^\omega$. Now for all $n \in \mathbb{N}$,

$$\begin{aligned}\min\{1, |x_n - z_n|\} &\leq \min\{1, |x_n - y_n| + |y_n - z_n|\} && \text{by the triangle inequality}\\ &\leq \min\{1, |x_n - y_n|\} + \min\{1, |y_n - z_n|\}\end{aligned}$$

And in particular,

$$\begin{aligned}\frac{\min\{1, |x_n - z_n|\}}{n} &\leq \frac{\min\{1, |x_n - y_n|\}}{n} + \frac{\min\{1, |y_n - z_n|\}}{n}\\ &\leq d(z,y) + d(y,z)\end{aligned}$$

And since this is true for all $n \in \mathbb{N}$, we conclude that

$$d(x,z) \leq d(x,y) + d(y,z)$$

Hence, d is a metric upon $\mathbb{R}^\omega$.

Let the metric topology induced by d over $\mathbb{R}^\omega$ be denoted by $\mathcal{T}_1$, and the product topology of $\mathbb{R}^\omega$ be denoted by $\mathcal{T}_2$. We wish to show that $\mathcal{T}_1 = \mathcal{T}_2$.

$\mathcal{T}_1 \subseteq \mathcal{T}_2$:

Let us take $x \in \mathbb{R}^\omega$ and some $U_1 \in \mathcal{T}_1$ such that $x \in U_1$. Since U_1 is open, we can find a ball $B^d_\epsilon(x) \subseteq U_1$ (for some $\epsilon > 0$).

Consider now the open 'box' $U_2 \in \mathcal{T}_2$ where $U_2 := (x_1 - \epsilon, x_1 + \epsilon) \times \ldots \times (x_N - \epsilon, x_N + \epsilon) \times \mathbb{R} \times \mathbb{R} \times \ldots$, where $N \in \mathbb{N}$ is large enough that $1/N < \epsilon$. This is certainly a valid basis element of $\mathcal{T}_2$, since we have the product of open sets in $\mathbb{R}$ with only finitely many of them being not equal to $\mathbb{R}$ itself.

Notice now that given any $y \in \mathbb{R}^\omega$,

$$\frac{\min\{1, |x_n - y_n|\}}{n} \leq \frac{1}{N} \qquad \text{for } n \geq N$$

Hence,

$$d(x,y) \leq \max\left\{\frac{\min\{1,|x_1-y_1|\}}{1}, \dots, \frac{\min\{1,|x_N-y_N|\}}{N}, \frac{1}{N}\right\}$$

And if $y \in U_2$, this expression is less than ϵ. So $U_2 \subseteq B_\epsilon^d(x) \subseteq U_1$. So we have proved that the metric topology is contained within the product topology.

$\underline{\mathcal{T}_2 \subseteq \mathcal{T}_1}$:

Again, we take $x \in \mathbb{R}^\omega$, and some $U_2 \in \mathcal{T}_2$ to be such that $x \in U_2$. Since U_2 is in the product topology of $\mathbb{R}^n$, we can find a $V \in \mathcal{T}_2$ defined $V := \prod_{n\in\mathbb{N}} V_n$ where V_n are open subintervals in $\mathbb{R}$ for $n \in \{\alpha_1, \dots, \alpha_N\}$ (for some $N \in \mathbb{N}$) and $V_n = \mathbb{R}$ for all other values of n, such that $x \in V$ (this V is a basis element of $\mathcal{T}_2$).

We now choose an interval $(x_n - \epsilon_n, x_n + \epsilon_n) \subseteq V_n \subseteq \mathbb{R}$ for $n \in \{\alpha_1, \dots, \alpha_N\}$, where each $\epsilon_n \leq 1$ (since any open set in $\mathbb{R}$ contains a sufficiently small open interval). This allows us to define,

$$\epsilon = \min\{\epsilon_n/n \mid n \in \{\alpha_1, \dots, \alpha_N\}\}$$

which certainly exists as we are finding the minimum over a finite domain.

Consider now $y \in B_\epsilon^d(x)$. Then for all $n \in \mathbb{N}$,

$$\frac{\min\{1,|x_n-y_n|\}}{n} \leq d(x,y) < \epsilon$$

Now if $n \in \{\alpha_1, \dots, \alpha_N\}$, we know that $\epsilon \leq \epsilon_n/n$. So $\min\{1,|x_n-y_n|\} < \epsilon_n \leq 1$. Hence $|x_n - y_n| < \epsilon$.

Therefore $B_\epsilon^d(x) \subseteq V \subseteq U_2$. So we have proved that the product topology is contained within the metric topology. □

Quotients

In the following exercise, we make use of:

Definition. A topological space is a pair $(X, \mathcal{T})$ where X is a set and $\mathcal{T}$ is a set of subsets of X, such that

(T1) $\emptyset, X$ both belong to $\mathcal{T}$

(T2) if $U, V \in \mathcal{T}$ then $U \cap V \in \mathcal{T}$

(T3) if $\{V_i\}_{i \in I}$ is any indexed set with $V_i \in \mathcal{T}$ for all $i \in I$, then $\bigcup_{i \in I} V_i \in \mathcal{T}$.

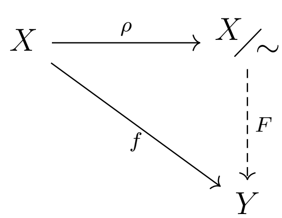
We call such a set $\mathcal{T}$ a topology on X and say that the sets $V \in \mathcal{T}$ are open in the topology. A set $C \subseteq X$ is closed in the topology if there exists $U \in \mathcal{T}$ with $C = X \setminus U$.

Definition. Let X be a topological space and $\sim$ be an equivalence relation on X. The quotient space $X/\sim$ is the set of equivalence classes with the topology given by ($\rho : X \to X/\sim$ denotes the quotient map)

$$\mathcal{T} := \left\{ U \subseteq X/\sim \;\middle|\; \rho^{-1}(U) \text{ is open in } X \right\}$$

Exercise L7-7

Prove this is a topology on $X/\sim$ and that for any space Y and for any continuous $f : X \to Y$ s.t. $f(x_1) = f(x_2)$ whenever $x_1 \sim x_2$, there is a unique continuous map F making the following diagram commute:



Proof (topology). To show that $\mathcal{T}$ is indeed a topology, we need to show that it satisfies the above definition.

(T1) $\emptyset \subseteq X/\sim$, and $\rho^{-1}(\emptyset) = \emptyset$, which is open in X. So $\emptyset \in \mathcal{T}$.
Similarly, $X/\sim \subseteq X/\sim$, and $\rho^{-1}\left(X/\sim\right) = X$, which is open in X. So $X/\sim \in \mathcal{T}$.

(T2) Suppose we have $U, V \in \mathcal{T}$. This means that $\rho^{-1}(U)$ and $\rho^{-1}(V)$ are open in X.

Now, suppose we have

$$\begin{aligned} & x \in \rho^{-1}(U \cap V) \\ \iff & \rho(x) \in U \cap V \\ \iff & \rho(x) \in U \text{ and } \rho(x) \in V \end{aligned}$$

$$\iff x \in \rho^{-1}(U) \text{ and } x \in \rho^{-1}(V)$$
$$\iff x \in \rho^{-1}(U) \cap \rho^{-1}(V)$$

Hence,

$$\rho^{-1}(U \cap V) = \rho^{-1}(U) \cap \rho^{-1}(V)$$

So $U \cap V \in \mathcal{T}$, since $\rho^{-1}(U \cap V)$ is the intersection of two open sets in X and is therefore open in X.

(T3) Suppose we have $\{U_i\}_{i\in I}$ is an indexed set with $U \in T$. This means that $\rho^{-1}(U_i)$ is open in X for all $i \in I$.

Now, suppose we have

$$\begin{aligned} & x \in \rho^{-1}\left(\bigcup_{i\in I} U_i\right) \\ \iff & \rho(x) \in \bigcup_{i\in I} U_i \\ \iff & \rho(x) \in U_i && \text{for some } i \in I \\ \iff & x \in \rho^{-1}(U_i) && \text{for some } i \in I \\ \iff & x \in \bigcup_{i\in I} \rho^{-1}(U_i) \end{aligned}$$

Hence,

$$\rho^{-1}\left(\bigcup_{i\in I} U_i\right) = \bigcup_{i\in I} \rho^{-1}(U_i)$$

So $\bigcup_{i\in I} U_i \in \mathcal{T}$, since $\rho^{-1}\left(\bigcup_{i\in I} U_i\right)$ is the union of arbitrary open sets in X and is therefore open in X.

So we conclude that $\mathcal{T}$ is indeed a topology on $X/\sim$. □

Proof (universal property). We first note that such an F is well-defined, as it is sending equivalence classes in $X/\sim$ to what f preserves in the equivalence relation, which are of course equal (by the definition of f).

Uniqueness:

Suppose that we have another continuous G such that $G \circ \rho = f$. Let $[x] \in X/\sim$ be the class of elements equivalent to $x \in X$ under $\sim$.

We notice that,

$$F([x]) = F(\rho(x)) = f(x) = G(\rho(x)) = G([x])$$

Which is to say that $F = G$ as functions, as every element of $X/\sim$ are of the form $[x]$ for some $x \in X$.

Existence: To show that such an F is continuous, we must show that the preimage of open subsets of Y map to open subsets of $X/\sim$.

Let $U \subseteq Y$ be an open subset. To show that $F^{-1}(U) \in \mathcal{T}$, we need to consider $\rho^{-1}(F^{-1}(U))$.

Now, suppose we have

$$\begin{aligned}
& x \in \rho^{-1}(F^{-1}(U)) \\
\iff & \rho(x) \in F^{-1}(U) \\
\iff & F(\rho(x)) \in U \\
\iff & f(x) \in U && \text{as } f = F \circ \rho \\
\iff & x \in f^{-1}(U)
\end{aligned}$$

Hence,

$$\rho^{-1}(F^{-1}(U)) = f^{-1}(U)$$

So $F^{-1}(U) \in \mathcal{T}$, since $\rho^{-1}(F^{-1}(U)) = f^{-1}(U)$, which is open in X, as f is continuous. We then conclude that F is continuous. □