Exercise L6-7

We must show that $\sim$ as defined in the question is reflexive, symmetric and transitive.

Reflexivity: Let $(U, f) \in S_x$, so that U is an open subset of X containing x and $f\colon U \to \mathbb{R}$ is a continuous function. By letting $W = U$, we see that W satisfies:

- $W \subseteq U \cap U$ since $W = U = U \cap U$
- W is open since $W = U$ and U is open
- $x \in W$ since $x \in U$ and $W = U$
- $f|_W = f|_W$.

So indeed there does exist W such that $W \subseteq U \cap U$ and W is open, $x \in W$ and $f|_W = f|_W$. Thus, $(U, f) \sim (U, f)$ and so $\sim$ is reflexive.

Symmetry: Let $(U, f), (V, g) \in S_x$. We have:

$$\begin{aligned}(U,f) \sim (V,g) &\iff \exists\, W \subseteq U \cap V \text{ s.t. } W \text{ is open, } x \in W \text{ and } f|_W = g|_W \\ &\iff \exists\, W \subseteq V \cap U \text{ s.t. } W \text{ is open, } x \in W \text{ and } g|_W = f|_W \\ &\iff (V,g) \sim (U,f).\end{aligned}$$

Hence $(U, f) \sim (V, g) \iff (V, g) \sim (U, f)$ and we conclude $\sim$ is symmetric.

Transitivity: Let $(U, f), (V, g), (W, h) \in S_x$ and suppose that $(U, f) \sim (V, g)$ and $(V, g) \sim (W, h)$. We have

$$\begin{aligned}(U,f) \sim (V,g) &\implies \exists\, A \subseteq U \cap V \text{ s.t. } A \text{ is open, } x \in A \text{ and } f|_A = g|_A \\ (V,g) \sim (W,h) &\implies \exists\, B \subseteq V \cap W \text{ s.t. } B \text{ is open, } x \in B \text{ and } g|_B = h|_B\end{aligned}$$

Let $C = A \cap B$. Then:

- $C \subseteq U \cap W$ since $C = A \cap B \subseteq A \subseteq U \cap V \subseteq U$ and $C = A \cap B \subseteq B \subseteq V \cap W \subseteq W$
- C is open since A and B are open, so $C = A \cap B$ is open by property (T2) of topological spaces
- $x \in C$ since $x \in A$ and $x \in B$ implies $x \in A \cap B = C$
- $f|_C = h|_C$ since ($C \subseteq A$ and $f|_A = g|_A$ implies $f|_C = g|_C$) and ($C \subseteq B$ and $g|_B = h|_B$ implies $g|_C = h|_C$), so that $f|_C = g|_C = h|_C$.

Hence there exists C such that $C \subseteq U \cap W$ and C is open, $x \in C$ and $f|_C = h|_C$. Thus, $(U, f) \sim (W, h)$, and we have proved that $\sim$ is transitive.

To conclude, $\sim$ is reflexive, symmetric and transitive so is an equivalence relation on S_x.

Exercise L6-8

We must show that $\sim$ as defined in the question (i.e. to denote Lipschitz equivalent) is reflexive, symmetric and transitive.

Reflexivity: Let d be a metric on X. For any $x, y \in X$, we have

$$1 \times d(x,y) \leq d(x,y) \leq 1 \times d(x,y)$$

so letting $h = 1$ and $k = 1$, we have $hd(x,y) \leq d(x,y) \leq kd(x,y)$ for all $x, y \in X$. Hence $d \sim d$ so $\sim$ is reflexive.

Symmetry: Let d_1, d_2 be metrics on X. We have

$$\begin{aligned}
d_1 \sim d_2 &\iff \exists\, h, k > 0 \text{ s.t. } hd_2(x,y) \leq d_1(x,y) \leq kd_2(x,y)\, \forall x, y \in X \\
&\iff \exists\, h, k > 0 \text{ s.t. } hd_2(x,y) \leq d_1(x,y) \text{ and } d_1(x,y) \leq kd_2(x,y)\, \forall x, y \in X \\
&\iff \exists\, h, k > 0 \text{ s.t. } d_2(x,y) \leq \frac{1}{h} d_1(x,y) \text{ and } \frac{1}{k} d_1(x,y) \leq d_2(x,y)\, \forall x, y \in X \\
&\iff \exists\, h, k > 0 \text{ s.t. } \frac{1}{k} d_1(x,y) \leq d_2(x,y) \leq \frac{1}{h} d_1(x,y)\, \forall x, y \in X \\
&\iff \exists\, h', k' > 0 \text{ s.t. } h'd_1(x,y) \leq d_2(x,y) \leq k'd_1(x,y)\, \forall x, y \in X \quad (h' = \frac{1}{k} \text{ and } k' = \frac{1}{h}) \\
&\iff d_2 \sim d_1
\end{aligned}$$

Hence $d_1 \sim d_2 \iff d_2 \sim d_1$ and we conclude $\sim$ is symmetric.

Transitivity: Let d_1, d_2, d_3 be metrics on X and suppose $d_1 \sim d_2$ and $d_2 \sim d_3$. We have

$$\begin{aligned}
d_1 \sim d_2 &\implies \exists\, h, k > 0 \text{ s.t. } hd_2(x,y) \leq d_1(x,y) \leq kd_2(x,y)\, \forall x, y \in X \\
d_2 \sim d_3 &\implies \exists\, h', k' > 0 \text{ s.t. } h'd_3(x,y) \leq d_2(x,y) \leq k'd_3(x,y)\, \forall x, y \in X
\end{aligned}$$

Let $h'' = hh'$ and $k'' = kk'$. Then $h'', k'' > 0$, and using the above we have, for all $x, y \in X$,

$$h''d_3(x,y) = hh'd_3(x,y) \leq hd_2(x,y) \leq d_1(x,y) \leq kd_2(x,y) \leq kk'd_3(x,y) = k''d_3(x,y).$$

Hence there does exist $h'', k'' > 0$ such that $h''d_3(x,y) \leq d_1(x,y) \leq k''d_3(x,y)$ for all $x, y \in X$, so $d_1 \sim d_3$. Thus $\sim$ is transitive.

To conclude, $\sim$ is reflexive, symmetric and transitive so is an equivalence relation.

Exercise L6-9

Let d_1, d_2 be metrics on X and suppose that $d_1 \sim d_2$. Let $\mathcal{T}_{d_1}$ and $\mathcal{T}_{d_2}$ be the topologies associated with (X, d_1) and (X, d_2), respectively. We want to show $\mathcal{T}_{d_1} = \mathcal{T}_{d_2}$, which we will do by showing that $\mathcal{T}_{d_1} \subseteq \mathcal{T}_{d_2}$ and $\mathcal{T}_{d_1} \supseteq \mathcal{T}_{d_2}$.

Part 1: First, we'll show that $\mathcal{T}_{d_1} \subseteq \mathcal{T}_{d_2}$. Let $U \in \mathcal{T}_{d_1}$, then $U \subseteq X$. Fix any $x \in U$. Then since $U \in \mathcal{T}_{d_1}$, there exists $\epsilon > 0$ such that

$$\{y \in X \mid d_1(x,y) < \epsilon\} \subseteq U. \tag{1}$$

Now, since $d_1 \sim d_2$, there exists $h, k > 0$ such that $hd_2(x,y) \leq d_1(x,y) \leq kd_2(x,y)\, \forall y \in X$, in particular, there exists $k > 0$ such that $d_1(x,y) \leq kd_2(x,y)$ for all $y \in X$. Note that this implies

$$\{y \in X \mid kd_2(x,y) < \epsilon\} \subseteq \{y \in X \mid d_1(x,y) < \epsilon\} \tag{2}$$

since for any $a \in \{y \in X \mid kd_2(x,y) < \epsilon\}$, we have $kd_2(x,a) < \epsilon$, which implies $d_1(x,a) \leq kd_2(x,a) < \epsilon$ so that $d_1(x,a) < \epsilon$ and hence $a \in \{y \in X \mid d_1(x,y) < \epsilon\}$.

Now let $\epsilon' = \epsilon/k$, then $\epsilon' > 0$ (since $\epsilon, k > 0$) and

$$\begin{aligned}
\{y \in X \mid d_2(x,y) < \epsilon'\} &= \{y \in X \mid d_2(x,y) < \epsilon/k\} && (\text{since } \epsilon' = \epsilon/k) \\
&= \{y \in X \mid kd_2(x,y) < \epsilon\} \\
&\subseteq \{y \in X \mid d_1(x,y) < \epsilon\} && (\text{by } (2)) \\
&\subseteq U. && (\text{by } (1))
\end{aligned}$$

So there does exist an $\epsilon' > 0$ such that

$$\{y \in X \mid d_2(x,y) < \epsilon'\} \subseteq U.$$

Since $x \in U$ was arbitrary, we have that for all $x \in U$ there exists $\epsilon' > 0$ such that $\{y \in X \mid d_2(x,y) < \epsilon'\} \subseteq U$, and together with $U \subseteq X$ this implies that $U \in \mathcal{T}_{d_2}$. So $U \in \mathcal{T}_{d_1}$ implies $U \in \mathcal{T}_{d_2}$, hence $\mathcal{T}_{d_1} \subseteq \mathcal{T}_{d_2}$.

Part 2: Showing that $\mathcal{T}_{d_1} \supseteq \mathcal{T}_{d_2}$ is similar to part 1. Indeed, simply notice that $d_1 \sim d_2$ implies $d_2 \sim d_1$ by symmetry, then use the same argument as part 1 but with 1s and 2s swapped wherever necessary.

To conclude, we have $\mathcal{T}_{d_1} \subseteq \mathcal{T}_{d_2}$ and $\mathcal{T}_{d_1} \supseteq \mathcal{T}_{d_2}$, which implies $\mathcal{T}_{d_1} = \mathcal{T}_{d_2}$ as required.

Exercise L6-10

To show that the metrics d_1, d_2, d_∞ on $\mathbb{R}^n$, where $n \in \mathbb{N}$, are all Lipschitz equivalent, it suffices to show $d_2 \sim d_1$ and $d_2 \sim d_\infty$ and the rest will follow from symmetry and transitivity.

Part 1 (show $d_2 \sim d_1$): We will show that for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$

$$n^{-1/2} d_1(\mathbf{x},\mathbf{y}) \leq d_2(\mathbf{x},\mathbf{y}) \leq d_1(\mathbf{x},\mathbf{y}). \tag{1}$$

Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, $\mathbf{y} = (y_1, y_2, \ldots, y_n)$. To prove the right-hand inequality of (1), notice that by expanding the LHS below, we get

$$\begin{aligned}
\left(\sum_{i=1}^n |x_i - y_i|\right)^2 &= \sum_{i=1}^n |x_i - y_i|^2 + 2 \sum_{1 \leq i < j \leq n} |x_i - y_i||x_j - y_j| \\
&\geq \sum_{i=1}^n |x_i - y_i|^2 \qquad (\text{since } |x_i - y_i|, |x_j - y_j| \geq 0 \text{ for all } i, j \in \{1, 2, \ldots, n\}) \\
&= \sum_{i=1}^n (x_i - y_i)^2.
\end{aligned}$$

Taking square roots (note both LHS and RHS are non-negative), we get

$$\sum_{i=1}^n |x_i - y_i| \geq \sqrt{\sum_{i=1}^n (x_i - y_i)^2} \quad \text{i.e.} \quad d_1(\mathbf{x},\mathbf{y}) \geq d_2(\mathbf{x},\mathbf{y}).$$

Now we'll prove the left-hand inequality of (1). For $i \in \{1, 2, \ldots, n\}$, let $a_i = |x_i - y_i|$. Then $a_i \geq 0$ for each i. We have

$$\sum_{1 \leq i < j \leq n} (a_i - a_j)^2 \geq 0 \qquad \text{(squares are non-negative)}$$

$$\implies \sum_{1 \leq i < j \leq n} a_i^2 - 2a_i a_j + a_j^2 \geq 0$$

$$\implies \sum_{1 \leq i < j \leq n} a_i^2 + a_j^2 \geq 2 \sum_{1 \leq i < j \leq n} a_i a_j$$

$$\implies (n-1) \sum_{i=1}^{n} a_i^2 \geq 2 \sum_{1 \leq i < j \leq n} a_i a_j$$

$$\implies n \sum_{i=1}^{n} a_i^2 \geq \sum_{i=1}^{n} a_i^2 + 2 \sum_{1 \leq i < j \leq n} a_i a_j = \left(\sum_{i=1}^{n} a_i \right)^2$$

noting that the second last line aboves follows since in the sum $\sum_{1 \leq i < j \leq n} a_i^2 + a_j^2$, each term of the form a_i^2 appears exactly $n-1$ times, once in each of $a_1^2 + a_i^2, a_2^2 + a_i^2, \ldots, a_{i-1}^2 + a_i^2, a_i^2 + a_{i+1}^2, \ldots, a_i^2 + a_n^2$. Anyway, by substituting back $a_i = |x_i - y_i|$, we get

$$n \sum_{i=1}^{n} |x_i - y_i|^2 \geq \left(\sum_{i=1}^{n} |x_i - y_i| \right)^2$$

$$\implies \sqrt{n \sum_{i=1}^{n} |x_i - y_i|^2} \geq \sum_{i=1}^{n} |x_i - y_i| \qquad \text{(note both sides were non-negative)}$$

$$\implies \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2} \geq n^{-1/2} \sum_{i=1}^{n} |x_i - y_i|$$

$$\implies d_2(\mathbf{x}, \mathbf{y}) \geq n^{-1/2} d_1(\mathbf{x}, \mathbf{y}).$$

Thus, we have proved (1) holds for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. It immediately follows that $d_2 \sim d_1$, as letting $h = n^{-1/2} > 0$ and $k = 1 > 0$, we see that these values satisfy $h d_1(\mathbf{x}, \mathbf{y}) \leq d_2(\mathbf{x}, \mathbf{y}) \leq k d_1(\mathbf{x}, \mathbf{y}) \, \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.

Part 2 (show $d_2 \sim d_\infty$): We will show that for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$

$$d_\infty(\mathbf{x}, \mathbf{y}) \leq d_2(\mathbf{x}, \mathbf{y}) \leq n^{1/2} d_\infty(\mathbf{x}, \mathbf{y}). \tag{2}$$

Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, $\mathbf{y} = (y_1, y_2, \ldots, y_n)$. Choose any $m \in \{1, 2, \ldots, n\}$ such that

$$|x_m - y_m| = \max\{|x_i - y_i| \mid 1 \leq i \leq n\} = d_\infty(\mathbf{x}, \mathbf{y}).$$

To prove the left-hand inequality of (2), notice that

$$d_2(\mathbf{x}, \mathbf{y}) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2} \geq \sqrt{(x_m - y_m)^2} = |x_m - y_m| = d_\infty(\mathbf{x}, \mathbf{y})$$

since $(x_i - y_i)^2 \geq 0$ for each $i \in \{1, 2, \ldots, n\}$. To prove the right-hand inequality of (2), notice that by our choice of m, $0 \leq |x_i - y_i| \leq |x_m - y_m|$ and thus $(x_i - y_i)^2 \leq (x_m - y_m)^2$ for all $i \in \{1, 2, \ldots, n\}$.

Hence

$$d_2(\mathbf{x},\mathbf{y}) = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2} \leq \sqrt{\sum_{i=1}^{n}(x_m - y_m)^2} = \sqrt{n(x_m - y_m)^2} = n^{1/2}|x_m - y_m| = n^{1/2} d_\infty(\mathbf{x},\mathbf{y}).$$

Thus we have proved (2) holds for all $\mathbf{x},\mathbf{y} \in \mathbb{R}^n$. It immediately follows that $d_2 \sim d_\infty$, as letting $h = 1 > 0$ and $k = n^{1/2} > 0$, we see that these values satisfy $hd_\infty(\mathbf{x},\mathbf{y}) \leq d_2(\mathbf{x},\mathbf{y}) \leq kd_\infty(\mathbf{x},\mathbf{y})\, \forall \mathbf{x},\mathbf{y} \in \mathbb{R}^n$.

Putting part 1 and part 2 together and using symmetry and transitivity of $\sim$, we see that d_1, d_2, d_∞ must all be Lipschitz equivalent. In particular this holds for when $n = 2$ (i.e. d_1, d_2, d_∞ are metrics on $\mathbb{R}^2$), and in this case we can conclude using Exercise 6-9 that $(\mathbb{R}^2, \mathcal{T}_{d_1}) = (\mathbb{R}^2, \mathcal{T}_{d_2}) = (\mathbb{R}^2, \mathcal{T}_{d_\infty})$.

Exercise L7-1

(i)

Let $(X, \mathcal{T})$ be a topological space and let $\mathcal{B}$ be a subset of $\mathcal{T}$. We want to prove that $\mathcal{B}$ is a basis for $\mathcal{T}$ if and only if every $U \in \mathcal{T}$ can be written as the union set of a subset $\mathcal{C} \subseteq \mathcal{B}$.

Part 1: We'll first show that if $\mathcal{B}$ is a basis for $\mathcal{T}$ then every $U \in \mathcal{T}$ can be written as the union set of a subset $\mathcal{C} \subseteq \mathcal{B}$. Consider any $U \in \mathcal{T}$. Since $\mathcal{B}$ is a basis for $\mathcal{T}$, for any $x \in U$ there exists $B_x \in \mathcal{B}$ with $x \in B_x \subseteq U$. Fix one such B_x for each $x \in U$. Now, we claim that U is the union set of $\mathcal{C} = \{B_x \mid x \in U\}$, i.e.

$$U = \bigcup_{x \in U} B_x.$$

To prove this, first notice that for any $x \in U$, we have $x \in B_x \subseteq \bigcup_{x \in U} B_x$ (where B_x is as chosen for x previously), which implies $x \in \bigcup_{x \in U} B_x$. Hence $U \subseteq \bigcup_{x \in U} B_x$. For the other inclusion, let $y \in \bigcup_{x \in U} B_x$, then $y \in B_x$ for some $x \in U$. But then $y \in B_x \subseteq U$, so $y \in U$. This proves that $\bigcup_{x \in U} B_x \subseteq U$. Putting these two inclusions together, we get $U = \bigcup_{x \in U} B_x$. To conclude, every $U \in \mathcal{T}$ can be written as the union set of a subset of $\mathcal{B}$.

Part 2: Now we'll show that if every $U \in \mathcal{T}$ can be written as the union set of a subset $\mathcal{C} \subseteq \mathcal{B}$, then $\mathcal{B}$ is a basis for $\mathcal{T}$. Consider any $U \in \mathcal{T}$. Then, U can be written as the union set of some subset $\mathcal{C} \subseteq \mathcal{B}$, i.e.

$$U = \bigcup_{B \in \mathcal{C}} B.$$

Consider any $x \in U$, then $x \in \bigcup_{B \in \mathcal{C}} B$ so $x \in B$ for some $B \in \mathcal{C}$. Also, $B \subseteq \bigcup_{B \in \mathcal{C}} B = U$ so actually we have $x \in B \subseteq U$. In summary, we have shown that for any $U \in \mathcal{T}$ and any $x \in U$, there exists $B \in \mathcal{B}$ with $x \in B \subseteq U$, which shows that $\mathcal{B}$ is a basis for $\mathcal{T}$.

(ii)

Let $(X, \mathcal{T})$ and $(Y, \mathcal{T}_Y)$ be topological spaces, let $\mathcal{B}$ be a basis for $\mathcal{T}$ and consider any function $f\colon (Y, \mathcal{T}_Y) \to (X, \mathcal{T})$.

If f is continuous then for all $U \subseteq X$, $U \in \mathcal{T} \implies f^{-1}(U) \in \mathcal{T}_Y$. Now, for any $B \in \mathcal{B}$, we have $B \subseteq X$ and $B \in \mathcal{B} \subseteq \mathcal{T}$, so continuity of f gives $f^{-1}(B) \in \mathcal{T}_Y$. So if f is continuous then $f^{-1}(B) \in \mathcal{T}_Y$ for all $B \in \mathcal{B}$.

For the other direction, suppose $f^{-1}(B) \in \mathcal{T}_Y$ for all $B \in \mathcal{B}$. Consider any $U \subseteq X$ such that $U \in \mathcal{T}$. Since $\mathcal{B}$ is a basis for $\mathcal{T}$, the result from part (i) of Exercise 7-1 tells us that U can be written in the form

$$U = \bigcup_{B \in \mathcal{C}} B$$

for some subset $\mathcal{C} \subseteq \mathcal{B}$. Then, $f^{-1}(U) = f^{-1}(\bigcup_{B\in\mathcal{C}} B)$. Now, notice that

$$\begin{aligned} y \in f^{-1}(\bigcup_{B\in\mathcal{C}} B) &\iff f(y) \in \bigcup_{B\in\mathcal{C}} B \\ &\iff f(y) \in B \text{ for some } B \in \mathcal{C} \\ &\iff y \in f^{-1}(B) \text{ for some } B \in \mathcal{C} \\ &\iff y \in \bigcup_{B\in\mathcal{C}} f^{-1}(B) \end{aligned}$$

so that $f^{-1}(\bigcup_{B\in\mathcal{C}} B) = \bigcup_{B\in\mathcal{C}} f^{-1}(B)$. But since $\mathcal{C} \subseteq \mathcal{B}$, we have that $f^{-1}(B) \in \mathcal{T}_Y$ for all $B \in \mathcal{C}$ so using property (T3) of topological spaces we get $\bigcup_{B\in\mathcal{C}} f^{-1}(B) \in \mathcal{T}_Y$. Thus, $f^{-1}(U) = f^{-1}(\bigcup_{B\in\mathcal{C}} B) = \bigcup_{B\in\mathcal{C}} f^{-1}(B) \in \mathcal{T}_Y$. So, we have shown that for any $U \subseteq X$, if $U \in \mathcal{T}$ then also $f^{-1}(U) \in \mathcal{T}_Y$. This shows that f is continuous, which completes our proof.

(iii)

Let (X, d) be a metric space. The associated topology is defined by

$$\mathcal{T} = \{U \subseteq X \mid \forall x \in U \, \exists \epsilon > 0 \text{ s.t. } B_\epsilon(x) \subseteq U\}.$$

Let $\mathcal{B} = \{B_\epsilon(x) \mid x \in X, \epsilon > 0\}$. Now, for any $U \in \mathcal{T}$ and $x \in U$, from the definition of $\mathcal{T}$ there exists $\epsilon > 0$ such that $B_\epsilon(x) \subseteq U$. Note also that $d(x, x) = 0 < \epsilon$ so $x \in B_\epsilon(x)$. So $B_\epsilon(x)$ satisfies $B_\epsilon(x) \in \mathcal{B}$ and $x \in B_\epsilon(x) \subseteq U$, and there exists at least one such $B_\epsilon(x)$ for any $U \in \mathcal{T}$ and $x \in U$. This shows that $\mathcal{B}$ is a basis for $\mathcal{T}$, as required.

Exercise L7-6

(i)

Let $\{X_i\}_{i\in I}$ be an indexed family of topological spaces. We'll first prove that the topology on $X = \coprod_{i\in I} X_i$ given by

$$\mathcal{T} = \{\coprod_{i\in I} U_i \mid U_i \subseteq X_i \text{ is open for each } i \in I\}$$

is indeed a topology by verifying each of the topology axioms (T1), (T2), (T3) in turn.

(T1): Let $U_i = \emptyset$ for each $i \in I$, then each U_i is open (by (T1) for the X_is) so $\emptyset = \coprod_{i\in I} \emptyset = \coprod_{i\in I} U_i \in \mathcal{T}$ (note that $\coprod_{i\in I} \emptyset = \{(i, u) \mid i \in I, u \in \emptyset\} = \emptyset$).

And now if we instead let $U_i = X_i$ for each $i \in I$, each U_i is open (again by (T1) for the X_is), so $X = \coprod_{i\in I} X_i = \coprod_{i\in I} U_i \in \mathcal{T}$.

(T2): Consider any $U, V \in \mathcal{T}$. Then $U = \coprod_{i\in I} U_i$ and $V = \coprod_{i\in I} V_i$ where $U_i, V_i \subseteq X_i$ are open for each $i \in I$. We have

$$\begin{aligned}
U \cap V &= \left(\coprod_{i\in I} U_i\right) \cap \left(\coprod_{i\in I} V_i\right) \\
&= \{(i,u) \mid i \in I, u \in U_i\} \cap \{(i,u) \mid i \in I, u \in V_i\} \\
&= \{(i,u) \mid i \in I, u \in U_i \text{ and } u \in V_i\} \\
&= \{(i,u) \mid i \in I, u \in U_i \cap V_i\} \\
&= \coprod_{i\in I} U_i \cap V_i
\end{aligned}$$

and since $U_i \cap V_i$ is open for each $i \in I$ (by (T2) for the X_is), we have $\coprod_{i\in I} U_i \cap V_i \in \mathcal{T}$. Hence $U \cap V = \coprod_{i\in I} U_i \cap V_i \in \mathcal{T}$, as required.

(T3): Consider $\{U_j\}_{j\in J}$ with $U_j \in \mathcal{T}$ for each $j \in J$. We may write each U_j as

$$U_j = \coprod_{i\in I} U_{j,i} = \{(i,u) \mid i \in I, u \in U_{j,i}\}$$

where $U_{j,i} \subseteq X_i$ is open for each $j \in J$, $i \in I$. Then,

$$\begin{aligned}
\bigcup_{j\in J} U_j &= \bigcup_{j\in J} \{(i,u) \mid i \in I, u \in U_{j,i}\} \\
&= \{(i,u) \mid i \in I, u \in U_{j,i} \text{ for some } j \in J\} \\
&= \{(i,u) \mid i \in I, u \in \bigcup_{j\in J} U_{j,i}\} \\
&= \coprod_{i\in I} \left(\bigcup_{j\in J} U_{j,i}\right)
\end{aligned}$$

and since $U_{j,i} \subseteq X_i$ is open for each $j \in J$ and $i \in I$, for all $i \in I$ we have $\bigcup_{j\in J} U_{j,i}$ is open (by (T3) for the X_is). Hence $\coprod_{i\in I} \left(\bigcup_{j\in J} U_{j,i}\right) \in \mathcal{T}$, so that $\bigcup_{j\in J} U_j = \coprod_{i\in I} \left(\bigcup_{j\in J} U_{j,i}\right) \in \mathcal{T}$ and we are done.

Continuity of $\iota_j \colon X_j \to \coprod_{i\in I} X_i$ sending x to (j,x): Consider any open subset $U \in \coprod_{i\in I} X_i$. To show that ι_j is continuous, where $j \in I$, we need to show that $\iota_j^{-1}(U)$ is open as a subset of X_j. By the definition of the discrete union topology on $\coprod_{i\in I} X_i$, we may write U as

$$U = \coprod_{i\in I} U_i = \{(i,u) \mid i \in I, u \in U_i\}$$

where $U_i \subseteq X_i$ is open for each $i \in I$. Then

$$\begin{aligned}
\iota_j^{-1}(U) &= \{x \in X_j \mid \iota_j(x) \in U\} \\
&= \{x \in X_j \mid (j,x) \in \{(i,u) \mid i \in I, u \in U_i\}\} \\
&= \{x \in X_j \mid (j,x) \in \{(j,u) \mid u \in U_j\}\} \\
&= \{x \in X_j \mid x \in U_j\} \\
&= U_j
\end{aligned}$$

which is open since U_i is open for all $i \in I$. Thus, we have shown that ι_j is continuous for each $j \in I$.

(ii)

We claim that for any topological space Y, the function $\Psi\colon \operatorname{Cts}(\coprod_{i\in I} X_i, Y) \to \prod_{i\in I} \operatorname{Cts}(X_i, Y)$ given by $\Psi(G) = (G \circ \iota_i)_{i\in I}$, where ι_i is as defined in part (i), is a bijection. Note that Ψ is well-defined since each $G \in \operatorname{Cts}(\coprod_{i\in I} X_i, Y)$ is continuous, ι_i is continuous for each $i \in I$ (as shown in part (i)) and composites of continuous functions are continuous, so $(G \circ \iota_i)_{i\in I} \in \prod_{i\in I} \operatorname{Cts}(X_i, Y)$.

(Ψ is injective): Suppose that $\Psi(G) = \Psi(F)$ for some $G, F \in \operatorname{Cts}(\coprod_{i\in I} X_i, Y)$. Then

$$\begin{aligned}
(G \circ \iota_i)_{i\in I} &= (F \circ \iota_i)_{i\in I} \\
\implies G \circ \iota_i &= F \circ \iota_i \quad \forall\, i \in I \\
\implies (G \circ \iota_i)(x) &= (F \circ \iota_i)(x) \quad \forall\, i \in I, x \in X_i \\
\implies G((i,x)) &= F((i,x)) \quad \forall i \in I, x \in X_i \\
\implies G(y) &= F(y) \quad \forall y \in \coprod_{i\in I} X_i \\
\implies G &= F
\end{aligned}$$

so Ψ is injective.

(Ψ is surjective): Given $(f_i)_{i\in I} \in \prod_{i\in I} \operatorname{Cts}(X_i, Y)$ (i.e. $f_i\colon X_i \to Y$ is continuous for each $i \in I$), define $F\colon \coprod_{i\in I} X_i \to Y$ as follows:

$$F((i,x)) = f_i(x) \in Y \quad \forall (i,x) \in \coprod_{i\in I} X_i.$$

We claim that F as defined above is continuous. Indeed, let $U \subseteq Y$ be any open subset of Y. Then

$$\begin{aligned}
F^{-1}(U) &= \{(i,x) \in \coprod_{i\in I} X_i \mid F((i,x)) \in U\} \\
&= \{(i,x) \in \coprod_{i\in I} X_i \mid f_i(x) \in U\} \\
&= \{(i,x) \in \coprod_{i\in I} X_i \mid x \in f_i^{-1}(U)\} \\
&= \{(i,x) \mid i \in I, x \in f_i^{-1}(U)\} \\
&= \coprod_{i\in I} f_i^{-1}(U).
\end{aligned}$$

But, since U is open and f_i is continuous for each $i \in I$, we have that $f_i^{-1}(U)$ is open for each $i \in I$. Then, $\coprod_{i\in I} f_i^{-1}(U)$ is open from the definition of open sets in the disjoint union topology. Hence $F^{-1}(U) = \coprod_{i\in I} f_i^{-1}(U)$ is open so we have shown that F is continuous. Then, notice that

$$\begin{aligned}
F(\iota_i(x)) = F((i,x)) &= f_i(x) \quad \forall i \in I, x \in X_i \\
\implies F \circ \iota_i &= f_i \quad \forall i \in I \\
\implies \Psi(F) = (F \circ \iota_i)_{i\in I} &= (f_i)_{i\in I}.
\end{aligned}$$

So, for any $(f_i)_{i\in I} \in \prod_{i\in I} \operatorname{Cts}(X_i, Y)$ there exists $F \in \operatorname{Cts}(\coprod_{i\in I} X_i, Y)$ such that $\Psi(F) = (f_i)_{i\in I}$, which shows that Ψ is surjective.

To conclude, $\Psi\colon \operatorname{Cts}(\coprod_{i\in I} X_i, Y) \to \prod_{i\in I} \operatorname{Cts}(X_i, Y)$ is well-defined, injective and surjective, so is a bijection between $\operatorname{Cts}(\coprod_{i\in I} X_i, Y)$ and $\prod_{i\in I} \operatorname{Cts}(X_i, Y)$ as required.